CHAPTER 7

AXIOM OF CHOICE AND ITS EQUIVALENT FORMS

● PROBLEM 7-1

Quote the axiom of choice and explain it for a finite family of sets.

SOLUTION:

AXIOM OF CHOICE

Let $\mathcal{F}$ be any non-empty family of non-empty sets. Then a function called a choice function exists

$$f:\mathcal{F} \to \bigcup_{A \in \mathcal{F}} A \tag{1}$$

such that for all $A \in \mathcal{F}$,

$$f(A) \in A \tag{2}$$

Suppose $\mathcal{F}$ is a set of phone books from different cities. It is possible to choose one and only one name from each book. The axiom of choice is trivial for $\mathcal{F}$ finite.

In this chapter we shall examine the axiom of choice and its three equivalent principles (the Hausdorff maximality principle, Zorn's lemma, and the well-ordering principle of Zermelo).

The axiom of choice is, indeed, an axiom that cannot be proven as a theorem using the classical axioms of mathematics.

● PROBLEM 7-2

Let

$$f: A \to B \tag{1}$$

denote onto mapping. Apply the axiom of choice to prove that a subset C of A exists, such that C and B are equipotent and hence card $A \geq$ card B = card C.

SOLUTION:

Suppose $B \neq \phi$ (if $B = \phi$, then $C = \phi$). Consider the family of sets

$$\{f^{-1}(y) : y \in B\} \tag{2}$$

which forms a partition of A because sets (2) are disjoint and

$$\bigcup_{y \in B} \{f^{-1}(y) : y \in B\} = A. \tag{3}$$

Note that f is onto.

According to the axiom of choice, the family of sets $\{f^{-1}(y) : y \in B\}$ has a set of representatives C, such that for each $y \in B$

$$\{f^{-1}(y) : y \in B\} \cap C \tag{4}$$

is a single element.

Restriction of f to C

$$f \mid C : C \to B \tag{5}$$

is one-to-one and onto. Therefore, sets C and B are equipotent. Hence

$$\text{card } A \geq \text{card } B = \text{card } C. \tag{6}$$

● PROBLEM 7–3

Let $\{A_i : i \in I\}$ represent a family of non-empty sets. The generalized Cartesian product $\underset{i \in I}{X} A_i$ of the family $\{A_i\}$ is the set of all functions

$$f : I \to \bigcup_{i \in I} A_i \tag{1}$$

such that

$$f(i) \in A_i \text{ for all } i \in I. \tag{2}$$

Prove that, if $I \neq \phi$, then $\underset{i \in I}{X} A_i$ is non-empty.

SOLUTION:

Set $\{A_i : i \in I\}$ is a non-empty set, whose elements are non-empty sets (i.e., for each $i \in I$, $A_i \neq \phi$). According to the axiom of choice, a function g, called a choice function, exists such that

$$g : \{A_i : i \in I\} \to \bigcup_{i \in I} A_i. \tag{3}$$

Then, we can define function f by

$$f(i) = g(A_i) \in A_i \tag{4}$$

for all $i \in I$, where

$$f : I \to \bigcup_{i \in I} A_i \tag{5}$$

Hence, if $I \neq \phi$, then

$$\underset{i \in I}{X} A_i \neq \phi. \tag{6}$$

1. Let A represent any set. Show that the relation $\{(a, b) : a = b\}$ is a preordering.

2. Show that in R, $\leq$ is a preordering and $<$ is not.

3. Is inclusion a preordering?

SOLUTION:

We shall start with

DEFINITION OF PREORDER

A relation ρ in a set A, $\rho \subset A \times A$, is called a preorder if it is reflexive and transitive, that is, if

1. $\forall a \in A : (a, a) \in \rho$

2. $\forall a, b \in A : (a, b) \in \rho \wedge (b, c) \in \rho \Rightarrow (a, c) \in \rho$

A set A with preorder is called a preordered set $(A, <)$. Preorder is usually denoted by $<$ ($a < b$ is read "a precedes b").

1. Relation $=$ is a preorder. It is reflexive because for each $a \in A$, $a = a$. It is transitive:

$$(a = b) \wedge (b = c) \Rightarrow a = c.$$

Hence $(A, =)$ is a preordered set.

2. For each $a \in R$, $a \leq a$. Also

$$(a \leq b) \wedge (b \leq c) \Rightarrow (a \leq c).$$

Therefore, $\leq$ is a preordering and $(R, \leq)$ is a preordered set. Of course, $a < a$ is not true and $<$ is not a preorder.

3. Consider the power set $P(X)$ (i.e., the set of all subsets of X). The relation $\subset$ is a preorder because for each $A, B, C \in P(X)$

$$A \subset A \text{ and } (A \subset B) \wedge (B \subset C) \Rightarrow A \subset C.$$

Preordering by inclusion is defined by

$$A < B \quad \text{iff} \quad A \subset B.$$

A family of sets preordered in this manner is called preordered by inclusion.

● PROBLEM 7-5

Let C represent the set of complex numbers. Show that $|z_1| \leq |z_1|$, $z_1, z_2 \in C$ defines a preordering.

SOLUTION:

Any complex number $z \in C$ can be represented by

$$z = x + iy$$

where x, y are real numbers. Then

$$|z| = \sqrt{x^2 + y^2}.$$

For every $z \in C$ $|z| \leq |z|$ because

$$\sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2}.$$

Hence relation $|\ldots| \leq |\ldots|$ is reflexive. Let $z_1, z_2, z_3 \in C$ and $|z_1| \leq |z_2|$ and $|z_2| \leq |z_3|$. Then

$$\sqrt{x_1^2 + y_1^2} \leq \sqrt{x_2^2 + y_2^2} \text{ and } \sqrt{x_2^2 + y_2^2} \leq \sqrt{x_3^2 + y_3^2}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$. Hence

$$\sqrt{x_1^2 + y_1^2} \leq \sqrt{x_3^2 + y_3^2}$$

or

$$|z_1| \leq |z_3|.$$

Relation $|\ldots| \leq |\ldots|$ is also transitive. Thus $|\ldots| \leq |\ldots|$ is a preorder and $(C, |\ldots| \leq |\ldots|)$ is a preordered set.

● PROBLEM 7-6

Let D denote the diagonal, $D \subset A \times A$. Show that $\rho \subset A \times A$ is a preorder, if and only if $D \subset \rho$ and $\rho \circ \rho = \rho$. The diagonal is defined by

$$D = \{(x, x) : x \in A\}.$$

SOLUTION:

We have to prove that

$$(\rho \text{ is a preorder}) \Leftrightarrow \begin{pmatrix} D \subset \rho \\ \rho \circ \rho = \rho \end{pmatrix}.$$

$\Rightarrow$ ρ is a preorder. For each $a \in A$, $(a, a) \in \rho$. Hence $D \subset \rho$. Furthermore

$$(a, b) \in \rho \wedge (b, c) \in \rho \Rightarrow (a, c) \in \rho$$

which is equivalent to $\rho \circ \rho = \rho$.

$\Leftarrow$ Since $D \subset \rho$ we have

$$\forall a \in A : (a, a) \in \rho.$$

The set $\rho \circ \rho$ is

$$\rho \circ \rho = \{(a, c) : a, c \in A, \exists b \in A \text{ such that } (a, b) \in \rho \wedge (b, c) \in \rho\}.$$

By assumption

$$\rho \circ \rho = \rho.$$

Hence

$$(a, b) \in \rho \wedge (b, c) \in \rho \Rightarrow (a, c) \in \rho.$$

● PROBLEM 7–7

Show that the relation $\leq$ defined on R^2 by

$$(a_1, a_2) \leq (b_1, b_2) \tag{1}$$

if and only if

$$a_1 \leq b_1 \text{ and } a_2 \leq b_2 \tag{2}$$

is a partial order relation on R^2.

SOLUTION:

We shall apply the following:

DEFINITION OF PARTIAL ORDER

A relation $\leq$ on a set A is called a partial order, if and only if $\leq$ is reflexive, transitive and antisymmetric on A, that is, if $a \leq b$ and $b \leq a$, then $a = b$. Set A with a partial order relation on A is denoted by $(A, \leq)$ and is called a partially ordered set.

Relation $\leq$ defined in (1) is reflexive,

$$(a, b) \le (a, b) \text{ because } a \le a \text{ and } b \le b. \tag{3}$$

It is also transitive.

Suppose

$$(a_1, b_1) \le (a_2, b_2) \quad \text{and} \quad (a_2, b_2) \le (a_3, b_3) \tag{4}$$

then

$$a_1 \le a_2 \quad \text{and} \quad a_2 \le a_3$$

$$b_1 \le b_2 \quad \text{and} \quad b_2 \le b_3$$

Hence

$$a_1 \le a_3 \quad \text{and} \quad b_1 \le b_3$$

or

$$(a_1, b_1) \le (a_3, b_3). \tag{5}$$

Finally we show that $\le$ is antisymmetric. Indeed, if

$$(a_1, b_1) \le (a_2, b_2) \quad \text{and} \quad (a_2, b_2) \le (a_1, b_1) \tag{6}$$

then

$$a_1 \le a_2 \quad \text{and} \quad a_2 \le a_1$$

$$b_1 \le b_2 \quad \text{and} \quad b_2 \le b_1.$$

Therefore,

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2 \quad \text{or} \quad (a_1, b_1) = (a_2, b_2). \tag{7}$$

● PROBLEM 7-8

Show that the relation defined in Problem 7-7 is not a total order.

SOLUTION:

We shall use

DEFINITION OF TOTAL ORDER

A total order relation $\le$ on a set A is a partial order relation, such that for any pair $a, b \in A$, either $a \le b$ or $b \le a$. Set A with a total order relation, denoted by $(A, \le)$, is called a totally ordered set.

Total order relation is sometimes called linear order relation. According

to the above definition, the relation described in Problem 7-7 is not a total order relation. Neither $(0, 1) \leq (1, 0)$, nor $(1, 0) \leq (0, 1)$.

This example shows that a partially ordered set need not be a totally ordered set.

● PROBLEM 7-9

Set $A = \{a, b, c\}$ is ordered as shown in the diagram.

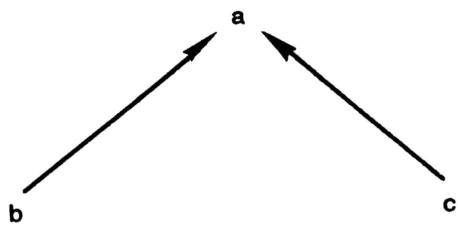


FIGURE 1

The order can be defined as follows: $x \leq y$ iff $x = y$ or one can move from x to y in the direction of arrows. Let B denote the collection of all non-empty totally ordered subsets of A. Draw the diagram of the order of B, when B is partially ordered by set inclusion.

SOLUTION:

The set of all subsets of A consists of

$$\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

Only the subsets $\{a, b, c\}$ and $\{b, c\}$ are not totally ordered, because neither $b \leq c$ nor $c \leq b$.

Set B consists of

$$\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\},$$

and is ordered by set inclusion. Hence

$$\{a\} \leq \{a, b\}$$

$$\{b\} \leq \{a, b\}$$

$$\{a\} \leq \{a, c\}$$

$$\{c\} \leq \{a, c\}$$

Set B is partially ordered as shown in the diagram

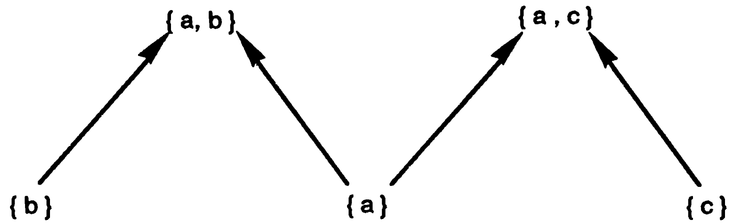


FIGURE 2

● PROBLEM 7-10

Consider the relation defined in Problem 7-7. Show that the diagonal of the plane R^2 is a chain.

SOLUTION:

Let A denote a partially ordered set $(A, \leq)$; then

$$P =: \{(x, y) : x, y \in A, x \leq y\}.$$

The sign =: will be used to denote "equals by definition."

For any subset B of A, $B \subset A$, we define $\leq'$ to be the intersection

$$P \cap (B \times B).$$

The set $(B, \leq')$ is a partially ordered set. It may happen that $(B, \leq')$ is a totally ordered subset of a partially ordered set.

DEFINITION OF A CHAIN

A totally ordered subset of a partially ordered set is called a chain.

In Problem 7-7, we showed that a relation defined by (1) is a partial order. The diagonal

$$D = \{(x, x) : x \in R\}$$

of the plane R^2 is a chain because on D, a relation becomes a total order.

● PROBLEM 7-11

Prove that the relation "*m* divides *n*" defined on the set of natural numbers N is a partial order, but not a total order.

SOLUTION:

A relation is reflexive, because for any natural number n, n divides n is true.

This relation is also transitive. Suppose m divides n and n divides k, that is,

$$\frac{n}{m} \in N \text{ , } \frac{k}{n} \in N$$

then

$$\frac{n}{m} \cdot \frac{k}{n} = \frac{k}{m} \in N.$$

That is, m divides k.

A relation is antisymmetric because, if m divides n and n divides m, then $m = n$. Hence, the relation is a partial order.

It is not a total order because

3 does not divide 2
and
2 does not divide 3.

● PROBLEM 7-12

Show that

1. The identity relation "=" is a partial order relation on any set.

2. The set $(R, \leq)$ is a totally ordered set, where R is the set of real numbers and "$\leq$" is understood in the ordinary sense.

SOLUTION:

1. Let A represent a set. Relation "=" is reflexive because for all $x \in A$

$$x = x. \tag{1}$$

It is also transitive. If

$$x = y \text{ and } y = z \text{, then } x = z. \tag{2}$$

Furthermore, relation "=" is antisymmetric. Hence, $(A, =)$ is a partially ordered set.

If set A contains only one element, then $(\{a\}, =)$ is a totally ordered set.

2. Relation "≤", understood in the ordinary sense and defined on the set of real numbers R, is a total order. For each $x \in R$

$$x \leq x.$$

If $x \leq y$ and $y \leq z$, then

$$x \leq z.$$

If $x \leq y$ and $y \leq x$, then

$$x = y.$$

For any pair of real numbers x, y, either $x \leq y$ or $y \leq x$.

● PROBLEM 7-13

Using the notion of temperature, define a total order relation on the set of all physical objects.

SOLUTION:

Temperature can be introduced in the following manner. Let a and b represent two objects. If there is a heat transfer from a to b, we say that the temperature of a is higher than the temperature of b. If there is no heat transfer, we say that a and b are of the same temperature. Here, we neglected all physical details.

Let us define the relationship $a \leq b$ iff the temperature of b is higher than or equal to the temperature of a. This relation is reflexive:

$$a \leq b.$$

It is transitive:

If $a \leq b$ and $b \leq c$, then

$$a \leq c.$$

It is antisymmetric:

If $a \leq b$ and $b \leq a$, then $a = b$ (that is, a and b have the same temperature).

Note that "temperature" defines the total order, since, for any two a and b, we have either $a \leq b$ or $b \leq a$.

● PROBLEM 7–14

Consider the set F of all functions

$$f: R \to R \tag{1}$$

with the relation S defined by

$$S = \{(f, g) \in F \times F : \forall x \in R, f(x) \le g(x)\}. \tag{2}$$

Prove that (F, S) is a partially ordered set.

SOLUTION:

Relation S is reflexive since

$$\forall f \in F \quad \forall x \in R \quad f(x) \le g(x). \tag{3}$$

S is transitive,

$$\begin{pmatrix} \forall f, g, h \in F, \forall x \in R \\ f(x) \le g(x) \\ g(x) \le h(x) \end{pmatrix} \Rightarrow \begin{pmatrix} f(x) \le h(x) \\ \forall x \in R \end{pmatrix}$$

S is antisymmetric,

$$\begin{pmatrix} f(x) \le g(x), \forall x \in R \\ g(x) \le f(x) \end{pmatrix} \Rightarrow \begin{pmatrix} f(x) = g(x) \\ \forall x \in R \end{pmatrix}$$

S is a partial order.

It is not a total order. For example, if $f(x) = x$ and $g(x) = x^2$, then neither $f(x) \le g(x)$ nor $g(x) \le f(x)$ for all $x \in R$.

● PROBLEM 7–15

Let A and B denote totally ordered sets $(A, \le')$, $(B, \le'')$. Show that $A \times B$ can be totally ordered.

SOLUTION:

We can define the order on $A \times B$ in this manner:

$$(a,b) \le (c,d) \left\langle \begin{array}{l} \text{if } a \le' c \\ \text{or, if } a = c \text{ and } b \le'' d \end{array} \right.$$

Since both $\le'$ and $\le''$ are reflexive, $\le$ is reflexive. Relation $\le$ is also transitive.

Suppose $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$. Then $a \leq' c$ and $c \leq' a$ imply $a = c$. Also $b \leq'' d$ and $d \leq'' b$ imply $b = d$.

Thus, $(a, b) = (c, d)$ and relation $\leq$ is antisymmetric.

Since $\leq'$ and $\leq''$ are total orders, $\leq$ is a total order.

The order defined above is called the lexicographical order on $A \times B$ because it is similar to the arrangement of the words in a dictionary.

● PROBLEM 7-16

Give an example of the partially ordered set that has no upper bound.

SOLUTION:

We shall start with

DEFINITION OF UPPER (LOWER) BOUND

Let $(A, \leq)$ denote a partially ordered set and B its subset, $B \subset A$. An element, $x \in A$, is an upper bound (lower bound) of B, if and only if

$$x \geq a \ (x \leq a) \text{ for all } a \in B.$$

For example, let $(N, \leq)$ denote a set of natural numbers $N = \{1, 2, 3, 4, \ldots\}$ and $B = \{2, 4, 8\}$ its subset. Relation "$\leq$" is an ordinary relation. Element $10 \in N$ is an upper bound of B. Any element of the set $\{8, 9, 10, 11, \ldots\}$ is an upper bound of B.

Elements 1 and 2 are lower bounds of B. Now consider the set of even numbers

$$\{2, 4, 6, 8, \ldots\}.$$

This set has no upper bound.

● PROBLEM 7-17

Let S represent a family of sets partially ordered by the relation of inclusion, and let R represent a subfamily of S.

1. Show that

$$\cup \{A : A \in R\}$$

does not have to be an upper bound of R.

2. Show that if $B \in S$ is an upper bound of R, then

$$\cup \{A : A \in R\} \subset B. \qquad (1)$$

SOLUTION:

1. Set S is a family of sets and R is its subfamily. It does not guarantee that the union of elements of R

$$\cup \{A : A \in R\}$$

is an element of S.

Hence, $\cup \{A : A \in R\}$ is an upper bound of R, if and only if it is an element of S.

2. Let

$$x \in \cup \{A : A \in R\}. \tag{2}$$

Then set A_0 exists, such that $x \in A_0$ and $A_0 \in R$.

Set B is an upper bound of R, therefore

$$A_0 \subset B \quad \text{and} \quad x \in B$$

which proves (1).

● PROBLEM 7-18

The set

$$A = \{a, b, c, d, e, f, g, h\} \tag{1}$$

is partially ordered as shown in the diagram.

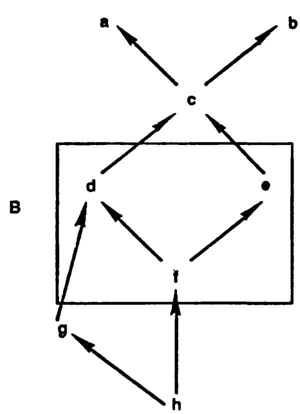


FIGURE 1

Two elements, x and y, are related, $x \leq y$, iff $x = y$ or one can move along the arrows from x to y. Consider the subset B of A

$$B = \{d, e, f\}.$$

Find the least upper bound and the greatest lower bound of B.

SOLUTION:

The elements a, b, and c are upper bounds of B.

DEFINITION OF LEAST UPPER BOUND (GREATEST LOWER BOUND)

Let $(A, \leq)$ represent a partially ordered set and B its subset. An upper bound (lower bound) x_0 of B is the least upper bound (the greatest lower bound) of B, if and only if $x_0 \leq x$ ($x_0 \geq x$) for every upper (lower) bound x of B.

Element c is the least upper bound of B because

$$c \leq a \quad \text{and} \quad c \leq b.$$

Elements f and h are lower bounds of B.

The greatest lower bound of B is f.

Sometimes, we denote the least upper bound of A by

$$\sup(A) \quad (\text{or } \operatorname{lub} A)$$

and the greatest lower bound of A by

$$\inf(A) \quad (\text{or } \operatorname{glb} A).$$

• PROBLEM 7-19

Let B denote a subset of the partially ordered set $(A, \leq)$. Prove that if the least upper bound (greatest lower bound) of B exists, then it is unique.

SOLUTION:

Proof (reductio ad absurdum or r.a.a.)

Suppose there are two least upper bounds of B, x and x', such that $x \neq x'$. Since x is the least upper bound of B and x' is an upper bound of B

$$x \leq x'. \tag{1}$$

Since x' is the least upper bound of B and x is an upper bound of B

$$x' \leq x. \tag{2}$$

Relation $\leq$ defined as A is a partial order. Hence, it is antisymmetric

$$\begin{pmatrix} x \leq x' \\ x' \leq x \end{pmatrix} \Rightarrow (x = x').$$

This is in contradiction with the assumption that $x \neq x'$. Thus, the least upper bound is unique.

In the same way, it can be shown that the greatest lower bound is unique.

● PROBLEM 7–20

Let X denote a non-empty set. The set $(P(X), \subset)$ is a partially ordered set. Show that, for any $Q \subset P(X)$, the least upper bound of Q is

$$\sup(Q) = \bigcup_{A \in Q} A \tag{1}$$

and the greatest lower bound of Q is

$$\inf(Q) = \bigcap_{A \in Q} A. \tag{2}$$

SOLUTION:

Let B represent any element of Q, $B \in Q$, then

$$B \subset \bigcup_{A \in Q} A. \tag{3}$$

Hence, $\bigcup_{A \in Q} A$ is the upper bound of Q. Suppose D is the upper bound of Q, then for any $B \in Q$

$$B \subset D. \tag{4}$$

Hence

$$\bigcup_{A \in Q} A \subset D \tag{5}$$

and $\bigcup_{A \in Q} A$ is the least upper bound.

The set $\bigcap_{A \in Q} A$ is the lower bound of Q because for any $B \in Q$

$$\bigcap_{A \in Q} A \subset B. \tag{6}$$

The set $\bigcap_{A \in Q} A$ is the greatest lower bound of Q. Indeed let D be the lower bound, then

$$D \subset B \text{ for all } B \in Q$$

and

$$D \subset \bigcap_{A \in Q} A. \tag{7}$$

● PROBLEM 7-21

The set

$$A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 14\}$$

is ordered by

$$x \leq y \text{ iff } x \text{ is a multiple of } y.$$

Find the maximal and minimal elements of A.

SOLUTION:

The order of A is depicted in the diagram

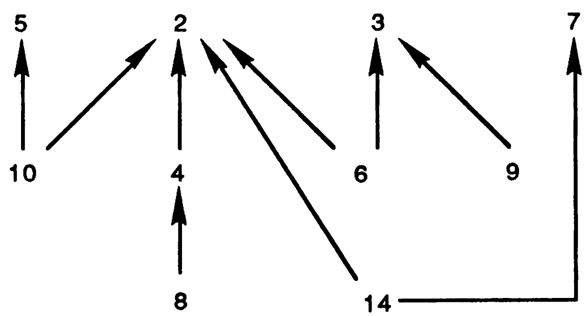


FIGURE 1

DEFINITION OF MAXIMAL (MINIMAL) ELEMENT

An element $x_0 \in A$ of a partially ordered set $(A, \leq)$ is said to be maximal (minimal), if and only if $x_0 \leq x$ $(x \leq x_0)$ implies $x = x_0$ for all $x \in A$.

According to the definition, the maximal elements of A are 2, 3, 5 and 7. The minimal elements are 6, 8, 9, 10, and 14.

● PROBLEM 7-22

Consider the set

$$A = \{2, 3, 4, 5, \ldots\} = N - \{1\}$$

ordered by $m \leq n$, iff m divides n.

Find maximal and minimal elements of A.

SOLUTION:

All prime numbers are minimal elements. If $p \in A$ is a prime number, then only p divides p, because $1 \notin A$.

Only prime numbers are minimal elements. Suppose $a \in A$ is not a prime number, then $b \in A$ exists, such that

$$b \text{ divides } a \ (b \leq a).$$

Hence, a is not a minimal element.

Set A has no maximal elements since, for every $m \in A$, $2m$ also belongs to A and

$$m \text{ divides } 2m \ (m \leq 2m).$$

● PROBLEM 7-23

Give an example of a partially ordered set that has more than one maximal element and more than one minimal element.

SOLUTION:

Consider set A consisting of elements

$$A = \{a, b, c, d, e\}$$

which are partially ordered as shown in the diagram.

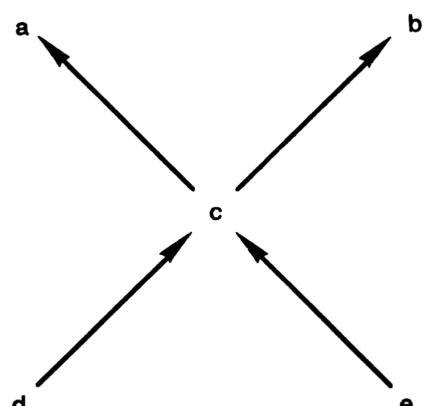


FIGURE 1

The maximal elements of A are a and b.

The minimal elements of A are d and e.

From this example, it is clear that maximal and minimal elements of a partially ordered set are not unique.

The situation changes for totally ordered sets (see Problem 7-24).

● PROBLEM 7–24

Show that if a totally ordered set has a maximal (or minimal) element, then this element is unique.

SOLUTION:

Set A is a totally ordered set $(A, \leq)$.

Suppose the set has two maximal elements, x_1 and x_2. We have $x_1 \in A$ and $x_2 \in A$. Since A is a totally ordered set for any pair $x_1, x_2 \in A$, either $x_1 \leq x_2$ or $x_2 \leq x_1$. Suppose $x_1 \leq x_2$. Then, since x_1 is a maximal element,

$$(x_1 \leq x_2) \Rightarrow (x_1 = x_2).$$

Hence, $x_1 = x_2$. The maximal element of a totally ordered set is unique.

In the same way, we can show that the minimal element of a totally ordered set is unique.

● PROBLEM 7–25

Draw the diagram for the ordered set (A, S) where

$$A = \{a, b, c, d, e, f, g\}$$

$$S = \{(b, c), (d, e), (e, f), (d, f), (g, a), (g, b), (g, c),$$

$$(g, d), (g, e), (g, f)\} \cup I_A.$$

Find the elements which are maximal, minimal, maximum, and minimum.

SOLUTION:

The diagram for the set (A, S) is shown in Figure 1.

The maximal elements are a, c, and f.

The minimal element is g.

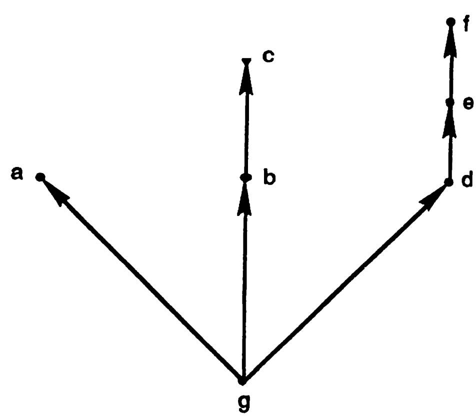


FIGURE 1

DEFINITION OF MAXIMUM (MINIMUM)

Element $a \in X$ of an ordered set X is a maximum (minimum) element of X or greatest (least) element of X, if and only if $x \le a$ $(a \le x)$ for all $x \in X$.

Set (A, S) has no maximum. Element g is a minimum element of A.

● PROBLEM 7-26

Show that if an ordered set A has two distinct minimal elements, then it has no minimum element.

SOLUTION:

Let x_1 and x_2 denote two distinct minimal elements of A. Suppose a is a minimum element of A. Therefore

$$a \le x_1 \qquad a \le x_2 \tag{1}$$

because a is a minimum element of A and $x_1, x_2 \in A$.

Since x_1 is a minimal element of A, there is no $x \in A$, such that

$$x_1 \ne x \le x_1. \tag{2}$$

From (1) and (2), we conclude

$$x_1 = a. \tag{3}$$

Similarly, since x_2 is a minimal element of A

$$x_2 = a. \tag{4}$$

Hence

$$x_1 = x_2 \tag{5}$$

which is a contradiction. Set A with two distinct minimal elements has no minimum element.

● PROBLEM 7-27

Let X represent a linearly ordered set and A a finite totally ordered subset of X. Show that if

$$a = \sup A$$

then

$$a = \text{maximum } A$$

Show that the conclusion can be false when A is not totally ordered.

SOLUTION:

Let

$$A = \{a_1, a_2, \ldots, a_n\} \tag{1}$$

and let

$$b_1 = a_1 = \text{maximum } \{a_1\}. \tag{2}$$

Moving step by step, we define

$$b_2 = \text{maximum } \{b_1, a_2\}$$

$$b_3 = \text{maximum } \{b_2, a_3\} \tag{3}$$

In general

$$b_i = \text{maximum } \{b_{i-1}, a_i\}$$

and

$$b_n = \text{maximum } \{b_{n-1}, a_n\}.$$

Set A is totally ordered, hence we can always find the maximum. Since the maximum exists,

$$a = \sup A = \text{maximum } A. \tag{4}$$

Suppose $x = \{a, b, c\}$ and $A = \{a, b\}$. Set A is not totally ordered; see Figure 1.

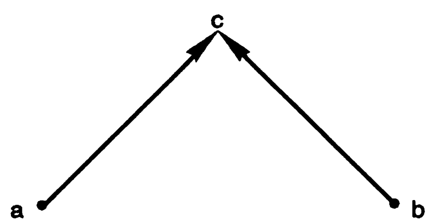


FIGURE 1

Then

$$c = \sup A \tag{5}$$

but c is not a maximum of A.

● PROBLEM 7-28

Prove that not every antisymmetric relation may be extended to an order.

SOLUTION:

Let X represent a set and T an antisymmetric relation on X. We have to choose T in such a way that no partial order Q on X exists, such that

$$T \subset Q. \tag{1}$$

Let

$$T = \{(a, b), (b, c), (c, d), (d, a)\} \tag{2}$$

denote an antisymmetric relation on X, where X is any set containing elements a, b, c, d.

Suppose a partial order Q exists, such that $T \subset Q$. Then

$$(a, b) \in Q \quad \text{and} \quad (b, c) \in Q. \tag{3}$$

Hence, since Q is transitive

$$(a, c) \in Q. \tag{4}$$

Furthermore

$$(a, d) \in Q \quad \text{and} \quad (d, a) \in Q \tag{5}$$

and, since Q is antisymmetric

$$a = d. \tag{6}$$

Similarly, we show that

$$a = b = c = d. \tag{7}$$

● PROBLEM 7-29

Let A and B represent ordered sets and let

$$f: A \rightarrow B \tag{1}$$

be onto. Prove that if for all $x, y \in A$

$$(x \leq y) \Leftrightarrow (f(x) \leq f(y)) \tag{2}$$

then f is one-to-one.

SOLUTION:

Suppose

$$f(x) = f(y) \tag{3}$$

Then

$$f(x) \geq f(y) \Rightarrow x \geq y$$

$$f(x) \leq f(y) \Rightarrow x \leq y. \tag{4}$$

Since order is a relation which is antisymmetric we conclude from (4) that

$$x = y. \tag{5}$$

● PROBLEM 7–30

We shall accept the following theorem without a proof.

THEOREM 1

Let $(A, \leq)$ represent a non-empty partially ordered set, such that every totally ordered subset of A has a least upper bound in A. If function $f : A \to A$ is such that for every $a \in A, f(a) \geq a$, then element $b \in A$ exists such that

$$f(b) = b.$$

Use Theorem 1 to prove:

THEOREM 2

Let $(A, \leq)$ represent a non-empty partially ordered set, such that every totally ordered subset of A has a greatest lower bound. If function $f : A \to A$ is such that for every $a \in A, f(a) \leq a$, then element $b \in A$ exists such that

$$f(b) = b.$$

SOLUTION:

Suppose (A, ρ) is a partially ordered set and ρ^{-1} is a relation inverse to ρ. Then (A, ρ^{-1}) is also a partially ordered set, where

$$(a \, \rho^{-1} \, b) \quad \text{iff} \quad (b \, \rho \, a). \tag{1}$$

Assume that (A, ρ) is such that every totally ordered subset of A has a least upper bound in A. Note that, if B is a totally ordered subset of (A, ρ), then (B, ρ^{-1}) is a totally ordered subset of (A, ρ^{-1}). If x_0 is a least upper bound of (B, ρ), then x_0 is a greatest lower bound of (B, ρ^{-1}).

Suppose function $f : A \to A$ is such that for every $a \in A, f(a) \geq a$. Then by applying Theorem 1, we conclude that element $b \in A$ exists, such that

$$f(b) = b.$$

If function f: $(A, \rho^{-1}) \to (A, \rho^{-1})$ is such that for every $a \in A$, $a \, \rho^{-1} f(a)$, then by virtue of Theorem 1, element $b \in A$ exists, such that

$$f(b) = b.$$

● PROBLEM 7–31

Show that the Hausdorff maximality principle and the axiom of choice are equivalent. Apply Theorem 1 of Problem 7-30.

SOLUTION:

HAUSDORFF MAXIMALITY PRINCIPLE

Let $(A, \leq)$ represent a partially ordered set and let S represent the set of all totally ordered subsets of A partially ordered by inclusion $(S, \subset)$. Then $(S, \subset)$ has a maximal element.

We shall use the reductio ad absurdum method.

Assume that S has no maximal element. Then, for every element s, $s \in S$, we can find a non-empty set

$$P = \{s' \in S : s \subset s', s \neq s'\}. \tag{1}$$

Elements s and s' of S are themselves ordered sets.

The axiom of choice guarantees the existence of a function f defined on $\{P : s \in S\}$ and such that

$$f(P) \in P. \tag{2}$$

Hence we can define a function

$$g : S \to S \tag{3}$$

such that

$$g(s) = f(P). \tag{4}$$

Note that since $g(s) = f(P)$, we have

$$s \subset g(s) = f(P), \quad g(s) \neq s \tag{5}$$

because $f(P) \in P$.

By applying Theorem 1 of Problem 7-30 [set $(S, \subset)$ and function g satisfy the hypotheses of the theorem], we conclude that an element $q \in S$ exists, such that

$$g(q) = q \tag{6}$$

which is a contradiction because for all $s \in S$

$$s \subset g(s) \quad (s \neq g(s)). \tag{7}$$

We showed that

(Axiom of Choice) $\Rightarrow$ (Hausdorff Maximality Principle).

Later we will carry out the proof in the opposite direction.

● PROBLEM 7-32

Use the Hausdorff maximality principle to prove the following:

Let $(A, \leq)$ represent a partially ordered set and B represent a totally ordered subset of A, $B \subset A$. Then A has a maximal totally ordered subset C such that

$$B \subset C. \tag{1}$$

SOLUTION:

Let S represent the set of all totally ordered subsets of A that contain B.

Note that the set $(S, \subset)$ is partially ordered.

The restriction that S consists of only these sets that contain B does not change the structure of the proof of theorem in Problem 7-31. Hence, we conclude that S has a maximal element, say C, such that

$$B \subset C.$$

● PROBLEM 7-33

Show that

(Axiom of Choice) $\Rightarrow$ (Zorn's Lemma).

SOLUTION:

ZORN'S LEMMA

If $(A, \leq)$ is a partially ordered set, such that every chain in A has an upper bound in A, then A contains a maximal element.

According to the Hausdorff maximality principle, the non-empty partially ordered set $(A, \leq)$ has a totally ordered subset $B \subset A$, which is maximal with respect to the relation of inclusion. By hypothesis, set B has an upper bound $x \in A$.

We shall show that element x is a maximal element of A.

Suppose, on the contrary, that an element $y \in A$ exists, such that

$$x \leq y.$$

Then $B \cup \{y\}$ is a totally ordered subset of A that contains B

$$B \subset B \cup \{y\}$$

where B is the maximal totally ordered subset of A. Hence

$$B \cup \{y\} = B$$

and

$$y \leq x.$$

Thus, x is a maximal element of $(A, \leq)$.

That completes the proof of

(Hausdorff Maximality Principle) $\Rightarrow$ (Zorn's Lemma).

From Problem 7-31, we obtain

(Axiom of Choice) $\Rightarrow$ (Hausdorff Maximality Principle) $\Rightarrow$

$\Rightarrow$ (Zorn's Lemma).

• PROBLEM 7–34

Prove that if set J of subsets of X has finite character, then $(J, \subset)$ has a maximal element.

SOLUTION:

DEFINITION OF FINITE CHARACTER

Set J of subsets of X has finite character, provided that $A \in J$, if and only if every finite subset of A belongs to J.

Set J is partially ordered by inclusion $(J, \subset)$.

Let $\mathcal{H}$ denote a chain in $(J, \subset)$. We shall show that $\mathcal{H}$ has an upper bound

$$B = \bigcup_{H \in \mathcal{H}} H$$

Let $\{x_1, x_2, \dots, x_n\}$ denote any finite subset of B,

$$\{x_1, x_2, \dots, x_n\} \subset \bigcup_{H \in \mathcal{H}} H.$$

Then

$$x_l \in H_l \text{ for some } H_l \in \mathcal{H}$$

$$l = 1, 2, \dots, n.$$

Set $\mathcal{H}$ is a chain, therefore $H \in \mathcal{H}$ exists, such that for all $l = 1, 2, \dots, n$

$$H_l \subset H.$$

Hence

$$\{x_1, x_2, \dots, x_n\} \subset H$$

and

$$\{x_1, x_2, \dots, x_n\} \in J.$$

Therefore $B \in J$ is an upper bound for $\mathcal{H}$.

According to Zorn's lemma, $(J, \subset)$ has a maximal element.

● PROBLEM 7–35

Apply Zorn's lemma to show that every vector space has a basis.

SOLUTION:

Let V denote a vector space, and let S denote the set of all linearly independent subsets of vectors in V.

Set S can be partially ordered by inclusion $(S, \subset)$. If H is a chain in $(S, \subset)$, then

$$T = \bigcup_{h \in H} h$$

is linearly independent. Hence T is an upper bound for H, $T \in S$.

According to Zorn's lemma, set $(S, \subset)$ has a maximal element. It can be easily shown that this element forms a basis for V.

● PROBLEM 7-36

Prove that every ring with an identity has a proper maximal ideal.

SOLUTION:

Let K denote a ring and let $1 \in K$ denote the identity.

Set $\mathcal{A}$ of all proper ideals of K is partially ordered by inclusion, $(\mathcal{A}, \subset)$. Let $(H, \subset)$ denote a chain in $(\mathcal{A}, \subset)$. Then

$$\bigcup_{h \in H} H$$

is a proper ideal of K.

Indeed

$$1 \notin \bigcup_{h \in H} h$$

Hence $\bigcup_{h \in H} h \in \mathcal{A}$ is an upper bound of H.

According to Zorn's lemma, we conclude that $(\mathcal{A}, \subset)$ has a maximal element. Hence, ring K with an identity has a proper maximal ideal.

● PROBLEM 7-37

Prove

(Zorn's lemma) $\Rightarrow$ (Kuratowski's lemma).

SOLUTION:

KURATOWSKI'S LEMMA

Each chain in a partially ordered set $(X, \leq)$ is included in a maximal chain.

Let $(X, \leq)$ denote a partially ordered set and h a given chain in X. Let H represent a set of all chains in X which include h, $h \in H$. Set H is ordered by inclusion $(H, \subset)$. Let a non-empty set $\mathcal{F} \neq \phi$ represent an inclusion chain of elements of H; then

$$\bigcup \mathcal{F} \in H. \tag{1}$$

Suppose $a, b \in \bigcup \mathcal{F}$; then for some $A, B \in \mathcal{F}$

$$a \in A \in \mathcal{F}$$

$$b \in B \in \mathcal{F}. \tag{2}$$

Since A and B are members of $\mathcal{F}$, we must have

$$A \subset B \quad \text{or} \quad B \subset A. \tag{3}$$

Hence

$$a \leq b \quad \text{or} \quad b \leq a$$

because $\cup \mathcal{F}$ is a chain. We have

$$h \subset A \subset \cup \mathcal{F} \in H. \tag{4}$$

Set $\mathcal{F}$ has an upper bound in H.

Therefore, according to Zorn's lemma, H has a maximal element.

Chain h is included in a maximal element.

One can prove that

(Kuratowski's Lemma) ⇒ (Zorn's Lemma).

Therefore, Zorn's lemma and Kuratowski's lemma are equivalent.

• PROBLEM 7–38

Show that the set of natural numbers $(N, \leq)$ is well ordered when $\leq$ is understood in the usual sense.

SOLUTION:

DEFINITION OF A WELL ORDERED SET

A totally ordered set $(A, \leq)$ is a well ordered set iff every non-empty subset B of A contains a minimal element; that is, if

$$\underset{\phi \neq B \subset A}{\forall} \quad \underset{b \in B}{\exists} \quad \underset{x \in B}{\forall} \quad b \leq x$$

Element b is called the least element of B and relation $\leq$ is called a well ordered relation.

The set of natural numbers N is well ordered because every non-empty subset of N contains the least element.

On the other hand, the set of rational numbers with the ordinary "less than or equal" $\leq$ relation is not well ordered. Not all subsets of R (for example, $\{-1, -2, -3, -4, \dots\}$, contain the least element.

• PROBLEM 7–39

Show that the set Q of rational numbers can be well ordered.

SOLUTION:

Set $(N, \leq)$ is well ordered. Set Q is denumerable, hence a bijection

$$f : Q \to N \tag{1}$$

exists.

Relation $\leq$ on Q can be introduced in the following manner:

$$\left(\underset{a,b \in Q}{\forall} \quad a \leq b \right) \Leftrightarrow \left(\begin{matrix} f(a) \leq f(b) \\ f(a), f(b) \in N \end{matrix} \right).$$

Now, since N is well ordered, so is Q.

In this proof, Q can be replaced by any denumerable set.

● PROBLEM 7-40

Let (X, T) denote a totally ordered set and $Y \subset X$. Prove that a is the minimum element of Y, if and only if

$$\mathrm{Im}(T \cap (\{a\} \times Y)) = Y. \tag{1}$$

SOLUTION:

Let $y \in Y$, then

$$y \in \mathrm{Im}(T \cap (\{a\} \times Y)). \tag{2}$$

Hence

$$(x, y) \in T \cap (\{a\} \times Y) \tag{3}$$

for some $x \in X$ if $(a, y) \in T$. Furthermore

$$(x, y) \in (T \cap (\{a\} \times Y)) = Y \tag{4}$$

if and only if

$$(a, y) \in T$$

for all $y \in Y$ and if and only if a is the minimum element of Y.

In 1904 Ernst Zermelo used the axiom of choice to prove the amazing well ordering principle.

WELL ORDERING PRINCIPLE

Every set can be well ordered.

Prove

$$(\text{Zorn's Lemma}) \Rightarrow (\text{Well Ordering Principle}).$$

SOLUTION:

Let A represent any set and $\tilde{A}$ a family of all well ordered sets $(A_0, \leq_0)$ which are subsets of A. Set $\tilde{A}$ can be partially ordered by the relation $\tilde{\leq}$ defined by:

$$(A_1, \leq_1) \ \tilde{\leq} (A_2, \leq_2) \tag{1}$$

iff

1. $A_1 \subset A_2$, and
2. $x, y \in A_1$ and $x \leq_1 y$ imply $x \leq_2 y$, and
3. $x \in A_2 - A_1$ implies $y \leq_2 x$ for all $y \in A_1$. Set $(\tilde{A}, \tilde{\leq})$ is partially ordered.

Now, let $\tilde{B}$ represent a totally ordered subset of $\tilde{A}$. To apply Zorn's lemma, we have to show that $\tilde{B}$ has an upper bound. Consider $\left(\bigcup_{A \in \tilde{B}} A, \leq^*\right)$ where $\leq^*$ is defined by:

$$x \leq^* y \text{ iff both } x, y \text{ belong to some } (A_0, \leq_0)$$

and $(A_0, \leq_0) \in \tilde{B}$, and $x \leq_0 y$.

It is easy to verify that $\leq^*$ is a total order relation on $\bigcup_{A \in \tilde{B}} A$. Furthermore, we will show that

$$\left(\bigcup_{A \in \tilde{B}} A, \leq^*\right) \tag{2}$$

is well ordered.

Let

$$\phi \neq C \subset \bigcup_{A \in \tilde{B}} A, \tag{3}$$

then $(A_0, \leq_0) \in \tilde{B}$ exists, such that

$$A_0 \cap C \neq \phi.$$

Set $(A_0, \leq_0)$ is well ordered, so $A_0 \cap C$ contains the unique least element of $A_0 \cap C$ denoted by a_0.

For any $x \in C$, $(A_1, \leq_1) \in \tilde{B}$ exists, such that

$$(A_0, \leq_0) \tilde{\leq} (A_1, \leq_1) \tag{4}$$

and $a_0, x \in A_1$.

We have $a_0 \leq x$ and $a_0 \leq^* x$. Hence a_0 is the least element for $(C, \leq^*)$.

Thus $\left(\bigcup_{A \in \tilde{B}} A, \leq^*\right)$ is well ordered.

According to Zorn's lemma, $(\tilde{A}, \tilde{\leq})$ has a maximal element $(A_1, \leq_1)$. We will show that $A = A_1$.

Indeed, suppose $A \neq A_1$; then $y \in A - A_1$ and relation $\leq_1$ on A_1 can be extended to $A_1 \cup \{y\}$ by

$$x \leq_1 y \text{ for all } x \in A_1.$$

We conclude that

$$(A_1, \leq_1) \tilde{\leq} (A_1 \cup \{y\}, \leq_1). \tag{5}$$

But $(A_1, \leq_1)$ is a maximal element, hence, a contradiction.

● PROBLEM 7–42

Prove that

(Well Ordering Principle) ⇒ (Axiom of Choice).

SOLUTION:

Let $\mathcal{A}$ represent a non-empty family of non-empty sets. According to the well ordering principle (see Problem 7-41) a total order relation $\leq$ exists, such that

$$\left(\bigcup_{A \in \mathcal{A}} A, \leq\right) \tag{1}$$

is a well ordered set.

Since (1) is well ordered, each $A \in \mathcal{A}$ contains a least element. Thus, we can define function f

$$f : \mathcal{A} \to \bigcup_{A \in \mathcal{A}} A \tag{2}$$

by

$$f(A) = \text{least element of } A. \tag{3}$$

Function f is the choice function of the axiom of choice.

● PROBLEM 7-43

Show that the following statements are equivalent:

1. The axiom of choice
2. Hausdorff maximality principle
3. Zorn's lemma
4. Kuratowski's lemma
5. The well ordering principle.

SOLUTION:

There is nothing left to prove. We will merely summarize the results of this chapter.

In Problem 7-31, we showed that

$$\text{(Axiom of Choice)} \Rightarrow \text{(Hausdorff Maximality Principle).}$$

In Problem 7-33 we proved

$$\text{(Hausdorff Maximality Principle)} \Rightarrow \text{(Zorn's Lemma).}$$

In Problem 7-41 we proved

$$\text{(Zorn's Lemma)} \Rightarrow \text{(Well Ordering Principle).}$$

To make the circle complete, we proved in Problem 7-42

$$\text{(Well Ordering Principle)} \Rightarrow \text{(Axiom of Choice).}$$

Hence

$$\text{(Axiom of Choice)} \equiv \text{(Zorn's Lemma)} \equiv$$

$$\equiv \text{(Well Ordering Principle)} \equiv \text{(Hausdorff Maximality Principle)} \equiv$$

$$\equiv \text{(Kuratowski's Lemma).}$$

For Kuratowski's lemma, see Problem 7-37.

Note that all these propositions are purely existential. They state the fact that "something" exists but do not offer the means of finding "it." For example, the well ordering principle states that the set of real numbers can be well ordered. So far it is not known *how* the set of real numbers can be well ordered.

• PROBLEM 7-44

Give an example of a segment of (N, ≤), where ≤ is the usual "less than or equal."

SOLUTION:

DEFINITION OF A SEGMENT

A segment of a totally ordered set $(A, \leq)$ is a subset S of A such that, if

$$x \in S, \text{ and } y \in A, \text{ and } y \leq x,$$

then $y \in S$. A proper segment of A is a segment which is a proper subset of A.

Set $\{1\}$ is a segment of N, where $N = \{1, 2, 3, 4, \ldots\}$ is the set of natural numbers.

Each set of the form

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \ldots$$

is a segment.

Also the set of natural numbers is a segment.

• PROBLEM 7-45

Show that any union or intersection of segments is a segment.

SOLUTION:

Let $(A, \leq)$ represent a totally ordered set, and let $\mathcal{F}$ represent a family of segments of A.

Suppose

$$x \in \bigcup_{S \in \mathcal{F}} S \tag{1}$$

and $y \in A$ where

$$y \leq x. \tag{2}$$

Element x belongs to some segment, say S_0 of A, $x \in S_0$. Since S_0 is a segment and $x \in S_0$, $y \in A$, $y \leq x$, we have

$$y \in S_0. \tag{3}$$

Hence,

$$y \in \bigcup_{S \in \mathcal{F}} S \tag{4}$$

and $\bigcup_{S \in \mathcal{F}} S$ is a segment of A.

Suppose

$$x \in \bigcap_{S \in \mathcal{F}} S \tag{5}$$

and $y \in A$, and $y \leq x$. Then for every $S \in \mathcal{F}$, $x \in S$ and since S is a segment, $y \in S$.

Hence, $y \in \bigcap_{S \in \mathcal{F}} S$.

Therefore, $\bigcap_{S \in \mathcal{F}} S$ is a segment.

● PROBLEM 7–46

Prove that if $(A, \leq)$ is a well ordered set, then:

1. All segments of a segment are again segments.

2. For each segment S of A, $S \neq A$, an element $x \in A$ exists, such that $S = A_x$, where

$$A_x = \{a \in A : a < x\}. \tag{1}$$

We use $a < x$ to indicate $a \leq x$ and $a \neq x$.

SOLUTION:

1. Let S represent a segment of A and P represent a segment of S. Suppose $x \in P$ and $y \in A$, and

$$y \leq x. \tag{2}$$

Since x belongs to the segment S, we have

$$y \in S.$$

Now from $x \in P$, $y \in S$ and $y \leq x$ we conclude that $y \in P$ because P is a segment of S. Thus, P is a segment of A.

2. Let S represent a segment of A, such that $S \neq A$. Then set $A - S$ is non-empty. This set has a least element, which we will denote by x.

Thus, as can be easily verified

$$A_x = S \quad \text{where} \quad A_x = \{a \in A : a < x\}. \tag{3}$$

Prove that if $(A, \leq)$ is a well ordered set and $\mathcal{F}$ a family of segments of A, such that

1. any union of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

2. if $A_x \in \mathcal{F}$, then $A_x \cup \{x\} \in \mathcal{F}$,

$$A_x = \{a \in A : a < x\}$$

then $\mathcal{F}$ contains all segments of A.

SOLUTION:

Suppose $x \in A$ exists, such that $A_x \notin \mathcal{F}$. Then, non-empty set

$$B = \{x \in A : A_x \notin \mathcal{F}\} \subset A \tag{1}$$

has a least element b, because $(A, \leq)$ is well ordered. We have

$$b \in B \quad \text{and} \quad A_b \notin \mathcal{F}. \tag{2}$$

If $y \in A$ and $y < b$, then

$$y \notin B \text{ and hence } A_y \notin \mathcal{F}.$$

By number 1, of the hypothesis,

$$\bigcup_{y<b} A_y \in \mathcal{F} \tag{3}$$

An element $a \in A$ exists (see Problem 7-46), such that

$$\bigcup_{y<b} A_y = A_a \in \mathcal{F} \tag{4}$$

hence $a < b$ and

$$A_a \cup \{a\} = A_c \text{ for some } c \in A. \tag{5}$$

Thus

$$a < c < b \quad \text{and}$$

$$a \in A_c \subset \bigcup_{y<b} A_y = A_a. \tag{6}$$

But $a \notin A_a$, hence a contradiction. Thus

$$\text{for all } x \in A, A_x \in \mathcal{F}. \tag{7}$$

Now, we shall prove that $A \in \mathcal{F}$. Indeed

$$\bigcup_{x \in A} A_x \in \mathcal{F} \tag{8}$$

if

$$A = \bigcup_{x \in A} A_x \, ; \tag{9}$$

then everything is all right. Suppose

$$A \neq \bigcup_{x \in A} A_x . \tag{10}$$

Then $y \in A$ exists, such that

$$\bigcup_{x \in A} A_x = A_y . \tag{11}$$

Furthermore

$$A_y \cup \{y\} \in \mathcal{F}. \tag{12}$$

Then

$$x \leq y \quad \text{for all} \quad x \in A.$$

Thus

$$A = A_y \cup \{y\} \in \mathcal{F}. \tag{13}$$

● PROBLEM 7–48

Prove the following important principle:

PRINCIPLE OF TRANSFINITE INDUCTION

Let $(A, \leq)$ denote a well ordered set and let $p(x)$ represent a sentence for each $x \in A$.

If, for each $x \in A$,

$$(p(y) \equiv 1 \text{ for every } y < x) \Rightarrow (p(x) \equiv 1) \tag{1}$$

then for every $x \in A, p(x) \equiv 1$.

Note that $p(x) \equiv 1$ denotes that the sentence $p(x)$ is true and $p(x) \equiv 0$ denotes that the sentence $p(x)$ is false.

SOLUTION:

Suppose $x \in A$ exists, such that

$$p(x) \equiv 0. \tag{2}$$

Let B denote a non-empty subset of A, such that

$$B = \{x \in A : p(x) \equiv 0\}, \tag{3}$$

and let b denote the least element of B. Remember that $(A, \leq)$ is well ordered.

Since $b \in B$, $p(b) \equiv 0$. Suppose $y \in A$, such that $y < b$; then $y \notin B$ and $p(y) \equiv 1$. Hence for every $y < b$

$$p(y) \equiv 1. \tag{4}$$

From (1) and (4), we conclude that

$$p(b) \equiv 1$$

which is a contradiction.

Thus

$$p(x) \equiv 1 \tag{5}$$

for every $x \in A$.

● PROBLEM 7-49

Apply the principle of transfinite induction to prove the following:

THEOREM

Let $(A, \leq)$ and $(A_0, \leq_0)$ denote well ordered sets. If $f : A \to A_0$ is increasing, $f(A)$ is a segment of A_0 and $F : A \to A_0$ is strictly increasing, then

$$\text{for all } x \in A, \quad f(x) \leq_0 F(x). \tag{1}$$

SOLUTION:

We shall start with

DEFINITION OF INCREASING FUNCTION

Let $(A, \leq)$ and $(A_0, \leq_0)$ denote well ordered sets. A function $f : A \to A_0$ is increasing iff

$$\begin{pmatrix} a \leq b \\ a, b \in A \end{pmatrix} \Rightarrow \begin{pmatrix} f(a) \leq_0 f(b) \\ f(a), f(b) \in A_0 \end{pmatrix}.$$

Function $f : A \to A_0$ is strictly increasing iff

$$\begin{pmatrix} a < b \\ a, b \in A \end{pmatrix} \Rightarrow \begin{pmatrix} f(a) < f(b) \\ f(a), f(b) \in A_0 \end{pmatrix}.$$

Let $p(x)$ represent the sentence

$$f(x) \leq_0 F(x).$$

Suppose an element $a \in A$ exists, such that for all $x < a$, $p(x) \equiv 1$ but $p(a) \equiv 0$. That is, for all $x < a$,

$$f(x) \leq_0 F(x) \quad \text{and} \quad F(a) <_0 f(a). \tag{2}$$

Function f is increasing and F is strictly increasing, thus for all $x < a$ and for all $a \leq y$

$$f(x) \leq_0 F(x) <_0 F(a) <_0 f(a) \leq_0 f(y). \tag{3}$$

Hence

$$F(a) <_0 f(a) \quad \text{and} \quad F(a) \notin f(A),$$

which is a contradiction because $f(A)$ is a segment of A_0. Hence, for each $x \in A$

$$\text{if } f(y) \leq_0 F(y) \text{ for all } y < x, \text{ then } f(x) \leq_0 F(x).$$

Now we can apply the principle of transfinite induction:

$$f(x) \leq_0 F(x) \text{ for all } x \in A.$$

CHAPTER 8

ORDINAL NUMBERS AND ORDINAL ARITHMETIC

• PROBLEM 8-1

Show that if the functions

$$f: A \to B$$

$$g: B \to C \tag{1}$$

are order isomorphic, then f^{-1} and $g \circ f$ are also order isomorphic.

SOLUTION:

DEFINITION OF ORDER ISOMORPHISM

Two well ordered sets $(A, \leq)$ and $(B, \leq_0)$ are order isomorphic if a bijection $f: A \to B$ exists, such that

$$\begin{pmatrix} a_1, a_2 \in A \\ a_1 \leq a_2 \end{pmatrix} \Rightarrow (f(a_1) \leq_0 f(a_2)). \tag{2}$$

Function $f: A \to B$ is called an order isomorphism.

■

If $f: A \to B$ is an order isomorphism, then so is $f^{-1}: B \to A$. Indeed, both sets A and B are well ordered. If f is a bijection, then f^{-1} is a bijection. Furthermore, if f has property (2), so does f^{-1}.

If functions f and g of (1) are bijections, then $g \circ f: A \to C$ is a bijection. Also if f and g have property (2), then $g \circ f$ has property (2).

• PROBLEM 8-2

Let $(A, \leq)$ and $(B, \leq_0)$ represent well ordered sets each consisting of the same finite number of elements. Show that $(A, \leq)$ and $(B, \leq_0)$ are order-isomorphic.

SOLUTION:

Sets $(A, \leq)$ and $(B, \leq_0)$ are equivalent, finite, and well ordered. Hence, their elements can be arranged as follows:

$$A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

$$B = \{\beta_1, \beta_2, \ldots, \beta_n\} \tag{1}$$

The arrangement $\alpha_1, \alpha_2, \ldots, \alpha_n$ is such that we denote the least element of

the set A by α_1; then we denote the least element of $A - \{\alpha_1\}$ by α_2, etc. The same procedure is repeated for the set B.

Bijection

$$f : A \to B$$

is defined by

$$f(\alpha_k) = \beta_k$$

for $k = 1, \ldots, n$.

Function f is an order isomorphism because if $\alpha_l \leq \alpha_k$, then $f(\alpha_l) \leq f(\alpha_k)$, where $\alpha_l \leq \alpha_k$ indicates that α_k appears after α_l in the sequence $\alpha_1, \ldots, \alpha_n$. We shall write

$$(A, \leq) \approx (B, \leq_0) \quad \text{or} \quad A \approx B$$

to indicate that the sets $(A, \leq)$ and $(B, \leq_0)$ are order isomorphic.

● PROBLEM 8-3

The set of natural numbers N is well ordered as follows.

$$(N, \leq) = \{1, 2, 3, 4, \ldots\} \tag{1}$$

and as

$$(N, \leq_0) = (1, 3, 5, \ldots, 2, 4, 6, \ldots\}. \tag{2}$$

Show that both sets are not order isomorphic.

SOLUTION:

It is easy to verify that both sets are well ordered. Suppose

$$f : (N, \leq) \to (N, \leq_0) \tag{3}$$

exists, such that f is order isomorphic. Then, to ensure that

$$\begin{pmatrix} a_1, a_2 \in (N, \leq) \\ a_1 \leq a_2 \end{pmatrix} \Rightarrow (f(a_1) \leq f(a_2))$$

we must have

$$f(1) = 1, f(2) = 3, f(3) = 5, \ldots$$

or in general

$$f(n) = 2n - 1. \tag{4}$$

Function (4) maps set N onto the set of odd numbers. Hence, (3) is not a bijection.

Thus, $(N, \leq)$ and $(N, \leq_0)$ are not order isomorphic.

● PROBLEM 8-4

Describe the concept of ordinal numbers.

SOLUTION:

1. To each well ordered set $(A, \leq)$, an ordinal number denoted by $\text{ord}(A, \leq)$, is assigned. Also, if α is an ordinal number, then a well ordered set $(A, \leq)$ exists, such that

$$\alpha = \text{ord}(A, \leq). \tag{1}$$

2. $\text{ord}(A, \leq) = \text{ord}(B, \leq_0)$ iff $(A, \leq) \approx (B, \leq_0)$

i.e., iff the sets $(A, \leq)$ and $(B, \leq_0)$ are order isomorphic.

3. $\text{ord}\,(A, \leq) = 0$ iff $A = \phi$.

4. If $(A, \leq)$ is a well ordered set consisting of n elements, then

$$\text{ord}\,(A, \leq) = n.$$

The ordinal number of the set N with the usual 1, 2, 3, ... order is denoted by ω. Thus

$$\text{ord}(N, \leq) = \omega.$$

A given set can have only one cardinal number, but depending on its order set, it can have many different ordinal numbers (for example see Problem 8-3).

● PROBLEM 8-5

Show that if $(A, \leq)$ is a well ordered set, then the only order isomorphism of $(A, \leq)$ onto a segment of $(A, \leq)$ is the identity function of A onto A.

SOLUTION:

Suppose, on the contrary, that an order isomorphism exists

$$f : A \to A_a \tag{1}$$

where A_a is a segment

$$A_a = \{x \in A : x < a\} \tag{2}$$

then $f(a) \in A_a$ and $f(a) < a$.

Let

$$B = \{x \in A : f(x) < x\} \neq \phi \tag{3}$$

and let b denote the least element of B. Then, since $b \in B$

$$f(b) < b \tag{4}$$

hence $f(b) \in B$, which is a contradiction because $f(b) < b$ and b is the least element of B. We conclude that a well ordered set cannot be order isomorphic to any of its proper segments. Now, suppose

$$f : A \to A \tag{5}$$

is order isomorphism.

Both functions f and the identity function $1_A : A \to$ A are strictly increasing. Hence,

$$1_A(x) \leq f(x) \leq 1_A(x) \tag{6}$$

for all $x \in A$ and $f = 1_A$.

● PROBLEM 8–6

> Let α and β denote two ordinal numbers. Show that the relation $\alpha \leq \beta$ (α is less than or equal to β) is reflexive and transitive.

SOLUTION:

We shall begin with the following:

DEFINITION

Let α and β denote ordinal numbers and let $(A, \leq)$ and $(B, \leq_0)$ denote well ordered sets, such that

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B, \leq_0). \tag{1}$$

Number α is less than or equal to β we denote it by $\alpha \leq \beta$, if and only if $(A, \leq)$ is order isomorphic to a segment of $(B, \leq_0)$. If $\alpha \leq \beta$ and $\alpha \neq \beta$, then we write $\alpha < \beta$.

■

Relation is reflexive if for every ordinal number α, $\alpha \leq \alpha$. Let $(A, \leq)$ represent a well ordered set, such that

$$\alpha = \text{ord}(A, \leq). \tag{2}$$

Set A is a segment of $(A, \leq)$. Identity function on A is an order isomorphism. Hence, $\alpha \leq \alpha$.

To prove that relation $\leq$ is transitive, we must show that for any ordinal numbers

$$(\alpha \leq \beta, \beta \leq \rho) \Rightarrow (\alpha \leq \rho). \tag{3}$$

Three well ordered sets exist, such that

$$\alpha = \mathrm{ord}(A, \leq), \ \beta = \mathrm{ord}(B, \leq_1), \ \rho = \mathrm{ord}(C, \leq_2).$$

If $(A, \leq)$ is order isomorphic to a segment of $(B, \leq_1)$ and $(B, \leq_1)$ is order isomorphic to a segment of $(C, \leq_2)$, then $(A, \leq)$ is order isomorphic to a segment of $(C, \leq_2)$.

● PROBLEM 8-7

Show that the relation $\leq$ for the ordinal numbers is a partial order relation.

SOLUTION:

A partial order relation is reflexive, transitive, and antisymmetric. We proved in Problem 8-5, that $\leq$ is reflexive and transitive. Now, we shall prove

THEOREM

$$\begin{pmatrix} \alpha, \beta \text{ are ordinal numbers} \\ \alpha \leq \beta \wedge \beta \leq \alpha \end{pmatrix} \Rightarrow (\alpha = \beta). \tag{1}$$

■

Let $(A, \leq)$ and $(B, \leq_0)$ represent well ordered sets, such that

$$\alpha = \mathrm{ord}(A, \leq), \ \ \beta = \mathrm{ord}(B \leq_0). \tag{2}$$

Since $\alpha \leq \beta$ and $\beta \leq \alpha$, two order isomorphisms exist, such that

$$f : A \to B_b \quad g : B \to A_a \tag{3}$$

where A_a and B_b are segments of A and B respectively. Function

$$h : A \to A_c \tag{4}$$

defined by

$$h(x) = g(f(x)), \text{ for all } x \in A \tag{5}$$

is an order isomorphism of A to a segment A_c of A_a. Since A_a is a segment of A, so is A_c. By Problem 8-5, $A_c = A$.

Thus, $A_a = A$ and since g is the order isomorphism $g : B \to A$, we conclude that

$$\alpha = \beta. \tag{6}$$

● PROBLEM 8-8

In Problem 8-3 we proved that

$$\operatorname{ord}(N, \le) \ne \operatorname{ord}(N, \le_0) \tag{1}$$

where

$$(N, \le) = \{1, 2, 3, 4, \dots\}$$

$$(N, \le_0) = \{1, 3, 5, \dots, 2, 4, 6, \dots\}. \tag{2}$$

Show that

$$\operatorname{ord}(N, \le) < \operatorname{ord}(N, \le_0). \tag{3}$$

SOLUTION:

Consider the function

$$f(n) = 2n - 1$$

$$f : (N, \le) \to (N, \le_0). \tag{4}$$

Let A denote the segment of $(N, \le_0)$

$$A = \{1, 3, 5, 7, \dots\}$$

Function $f(n) = 2n - 1$

$$f : (N, \le) \to A \tag{5}$$

is a bijection, which preserves order. Thus, f defined by (5) is an order isomorphism. Therefore,

$$\operatorname{ord}(N, \le) \le \operatorname{ord}(N, \le_0). \tag{6}$$

From (6) and (1), we obtain

$$\operatorname{ord}(N, \le) < \operatorname{ord}(N, \le_0). \tag{7}$$

● PROBLEM 8-9

Is the relation $\leq$, defined on ordinal numbers, a total order relation?

SOLUTION:

In Problem 8-7 we proved that relation $\leq$ is reflexive, transitive, and antisymmetric. That is, $\leq$ is a partial order.

We shall quote the following without a proof.

THEOREM

For any two cardinal numbers α and β, either

$$\alpha \leq \beta \quad \text{or} \quad \beta \leq \alpha.$$

■

Thus, for any two ordinal numbers, their order is determined.

We conclude that $\leq$ is a total order and ordinal numbers are totally ordered.

The above results can be summarized in the following theorem.

THEOREM

If α and β are ordinal numbers, then one and only one of the following is true:

1. $\alpha < \beta$

2. $\alpha = \beta$

3. $\beta < \alpha$.

■

● PROBLEM 8-10

Let $(A, \leq)$ and $(B, \leq_0)$ represent well ordered sets. Prove that

$$\left(\text{ord}(A,\leq) < \text{ord}(B,\leq_0)\right) \Leftrightarrow \left(\begin{array}{c}A \text{ is order isomorphic} \\ \text{to a proper segment of } B\end{array}\right).$$

SOLUTION:

$\Leftarrow$ If A is order isomorphic to a segment of B, then

$$\operatorname{ord}(A, \leq) \leq \operatorname{ord}(B, \leq_0). \tag{1}$$

But A is order isomorphic to a proper segment of B. From (1) and the theorem of Problem 8-5, we conclude that

$$\operatorname{ord}(A, \leq) < \operatorname{ord}(B, \leq_0). \tag{2}$$

$\Rightarrow$ Condition (2) can be written as

$$\operatorname{ord}(A, \leq) \leq \operatorname{ord}(B, \leq_0) \tag{3}$$

and

$$\operatorname{ord}(A, \leq) \neq \operatorname{ord}(B, \leq_0). \tag{4}$$

From (3), we conclude that A must be order isomorphic to a segment of B; and from (4) and theorem of Problem 8-5, we conclude that A is order isomorphic to a proper segment of B.

● PROBLEM 8–11

Show that

$$\operatorname{ord}\{k, k+1, k+2, \ldots\} = \operatorname{ord}\{1, 2, \ldots\} \tag{1}$$

where k is any natural number.

SOLUTION:

Let us define function

$$f: \{1, 2, 3, \ldots\} \to \{k, k+1, k+2, \ldots\} \tag{2}$$

by

$$f(n) = n + k - 1 \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{3}$$

Function f is a bijection and such that if $n_1 \leq n_2$ where $n_1, n_2 \in \{1, 2, 3, \ldots\}$, then

$$f(n_1) \leq f(n_2) \text{ where } f(n_1), f(n_2) \in \{k, k+1, \ldots\}.$$

Thus, f is an order isomorphism and

$$\{k, k+1, k+2, \ldots\} \approx \{1, 2, 3, 4, \ldots\} \tag{4}$$

which proves (1).

● PROBLEM 8-12

Prove that

$$\text{ord}\,\{1, 2, 3, \ldots\} < \text{ord}\,\{k, k+1, k+2, \ldots, 1, 2, 3, \ldots, k-1\}. \quad (1)$$

SOLUTION:

In other words, we have to prove that $\{1, 2, 3, \ldots\}$ is order isomorphic to a proper segment of $\{k, k+1, \ldots, 1, 2, 3, \ldots, k-1\}$.

The subset

$$\{k, k+1, k+2, \ldots\} \quad (2)$$

is a proper segment of

$$\{k, k+1, k+2, \ldots, 1, 2, \ldots, k-1\}. \quad (3)$$

In Problem 8-11, we proved that (2) is order isomorphic to

$$\{1, 2, 3, 4, \ldots\}.$$

That completes proof of (1).

● PROBLEM 8-13

What is the smallest transfinite ordinal number?

SOLUTION:

The ordinal number of the set consisting of n elements is n, assuming the set is well ordered. The infinite set with the lowest cardinal number is the set of natural numbers or any set equivalent to it.

A given set has only one cardinal number, but a set may have different ordinal numbers under distinct well orderings. The ordinal number of the set of natural numbers ordered as

$$\{1, 2, 3, 4, \ldots\}$$

is denoted by ω. Hence,

$$\omega = \text{ord}\,(N, \leq).$$

Any other ordering of N leads to the same or larger ordinal number. See, for example, Problem 8-12.

Hence, the smallest transfinite ordinal number is ω.

● PROBLEM 8–14

Find the ordinal sum

$$3 + 4$$

of the two finite ordinal numbers 3 and 4.

SOLUTION:

DEFINITION OF SUM

The ordinal sum $\alpha + \beta$ of ordinal numbers α and β is the ordinal number of the set $(A \cup B, \leq')$, where $(A, \leq)$ and $(B, \leq_0)$ are disjoint well ordered sets, such that

$$\alpha = \text{ord}(A, \leq) \quad \text{and} \quad \beta = \text{ord}(B, \leq_0).$$

Relation $\leq'$ is defined by

1. If $a, b \in A$ (or $a, b \in B$), then we write $a \leq' b$ iff $a \leq b$ (or $a \leq_0 b$).

2. If $a \in A$ and $b \in B$, we write $a \leq' b$.

■

Since

$$3 = \text{ord}\{1, 2, 3\}$$

and

$$4 = \text{ord}\ \{4, 5, 6, 7\}$$

we have

$$3 + 4 = \text{ord}\ \{1, 2, 3, 4, 5, 6, 7\} = 7.$$

● PROBLEM 8–15

Consider the definition of the sum of ordinal numbers given in Problem 8-14. Explain why the sets

$$(A \cup B, \leq') \text{ and } (B \cup A, \leq')$$

should be considered distinct.

SOLUTION:

To find the sum of ordinal numbers α and β, we take two well ordered

disjoint sets A and B, such that

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B, \leq_0).$$

If the sets are not disjoint we can replace them by $A \times \{x\}$ and $B \times \{y\}$. Then

$$A \approx A \times \{x\} \quad \text{and} \quad B \approx B \times \{y\}.$$

In the next step, we form the sum of the sets A and B and define well ordering, $(A \cup B, \leq')$, such that:

$$\text{if } a \in A \quad \text{and} \quad b \in B, \text{ we write } a \leq' b.$$

Hence, any element $a \in A$ is less than any element $b \in B$. While, for the sum $(B \cup A, \leq')$, any element $b \in B$ is less than any element $a \in A$, $b \leq' a$.

To avoid this whole problem, one can use different notation, such as $(A \cup B, \underset{A,B}{\leq'})$ meaning "A is less than B." Then

$$(A \cup B, \underset{A,B}{\leq'}) = (B \cup A, \underset{A,B}{\leq'}).$$

● PROBLEM 8-16

Let $k \neq 0$ denote any finite ordinal number. Show that

$$k + \omega = \omega \tag{1}$$

$$\omega + k \neq \omega. \tag{2}$$

SOLUTION:

We have

$$k = \text{ord}\,\{1, 2, \ldots, k\} \tag{3}$$

$$\omega = \text{ord}\,\{k+1, k+2, \ldots\} \tag{4}$$

(see Problem 8-11.) Thus,

$$k + \omega = \text{ord}\{1, 2, \ldots, k, k+1, \ldots\} = \omega \tag{5}$$

Similarly, we obtain

$$\omega + k = \text{ord}\,\{k+1, k+2, \ldots, 1, 2, 3, \ldots\} > \omega \tag{6}$$

(see Problem 8-12.) Thus,

$$\omega + k \neq \omega. \tag{7}$$

In general, addition of ordinal numbers is not communicative.

It is rather amazing that

$$\omega + k \neq k + \omega. \tag{8}$$

• PROBLEM 8-17

Let (A1, $\le_1$), (A_2, $\le_2$), (B_1, $\le_1'$), and (B_2, $\le_2'$) represent well ordered sets, such that

$$A_1 \cap B_1 = A_2 \cap B_2 = \phi \text{ and}$$

$$A_1 \approx A_2, \quad B_1 \approx B_2. \tag{1}$$

Prove that

$$(A_1 \cup B_1, \le_1^*) \approx (A_2 \cup B_2, \le_2^*). \tag{2}$$

SOLUTION:

Since $A_1 \approx A_2$ and $B_1 \approx B_2$, we have: bijection $f : A_1 \to A_2$ exists, such that

$$(a_1 \le_1 a_2) \Rightarrow (f(a_1) \le_2 f(a_2)) \tag{3}$$

and bijection $g : B_1 \to B_2$ exists, such that

$$(b_1 \le_1' b_2) \Rightarrow (g(b_1) \le_2' g(b_2)). \tag{4}$$

Functions f and g are order isomorphisms. We define

$$F : A_1 \cup B_1 \to A_2 \cup B_2 \tag{5}$$

by

$$F(x) = \begin{cases} f(x) \text{ if } x \in A_1 \\ g(x) \text{ if } x \in B_1. \end{cases} \tag{6}$$

Remember that $A_1 \cap B_1 = A_2 \cap B_2 = \phi$. It is easy to verify that F is a bijection. We define well orderings $\le_1^*$ and $\le_2^*$ according to the definition of Problem 8-14.

$$\text{if } x_1 \le_1^* x_2, \text{ then } \begin{array}{l} x_1, x_2 \in A_1,\ f(x_1) = F(x_1) \\ \qquad\qquad\quad\ f(x_2) = F(x_2) \\ \text{or} \downarrow \\ x_1, x_2 \in B_1,\ g(x_1) = F(x_1) \\ \qquad\qquad\quad\ g(x_2) = F(x_2) \\ \text{or} \downarrow \\ x_1 \in A_1, x_2 \in B_1 \\ F(x_1) = f(x_1) \in A_2 \\ F(x_2) = g(x_2) \in B_2 \end{array} F(x_1) \le_2^* F(x_2) \tag{7}$$

Thus, F defined in (4) and (5) is an order isomorphism and

$$(A_1 \cup B_1, \le_1^*) \approx (A_2 \cup B_2, \le_2^*). \tag{8}$$

Prove that ordinal sum is associative; that is, for any ordinals α, β, and ρ

$$(\alpha + \beta) + \rho = \alpha + (\beta + \rho). \tag{1}$$

SOLUTION:

It is possible to find three pair-wise disjoint sets A, B, and C, such that

$$\alpha = \operatorname{ord}(A, \leq_1)$$

$$\beta = \operatorname{ord}(B, \leq_2)$$

$$\rho = \operatorname{ord}(C, \leq_3). \tag{2}$$

According to the definition of Problem 8-14, we obtain

$$\alpha + \beta = \operatorname{ord}(A \cup B, \leq_1{}^*) \tag{3}$$

and

$$\beta + \rho = \operatorname{ord}(B \cup C, \leq_2{}^*) \tag{4}$$

where well ordering $\leq_1{}^*$ means "A is less than B" and well ordering $\leq_2{}^*$ means "B is less than C." We obtain

$$(\alpha + \beta) + \rho = \operatorname{ord}(A \cup B \cup C, \leq_3{}^*)$$

$$\alpha + (\beta + \rho) = \operatorname{ord}(A \cup B \cup C, \leq_4{}^*) \tag{5}$$

but $\leq_3{}^*$ and $\leq_4{}^*$ are exactly the same orders; i.e., "A is less than B" and "B is less than C." Thus,

$$(\alpha + \beta) + \rho = \alpha + (\beta + \rho) = \alpha + \beta + \rho =$$

$$= \operatorname{ord}(A \cup B \cup C, \leq^*). \tag{6}$$

If $x_1, x_2 \in A$, we write $x_1 \leq^* x_2$ iff $x_1 \leq_1 x_2$.

If $x_1, x_2 \in B$, we write $x_1 \leq^* x_2$ iff $x_1 \leq_2 x_2$.

If $x_1, x_2 \in C$, we write $x_1 \leq^* x_2$ iff $x_1 \leq_3 x_2$.

We write $x_1 \leq^* x_2$ if $x_1 \in A$ and $x_2 \in B \cup C$, or if $x_1 \in B$ and $x_2 \in C$.

● PROBLEM 8-19

Prove that

$$(\beta < \rho) \Rightarrow (\alpha + \beta < \alpha + \rho) \tag{1}$$

for any ordinal numbers α, β, and ρ.

SOLUTION:

Let $(A, \leq)$, $(B, \leq_1)$, and $(C, \leq_2)$ represent well ordered sets, such that

$$\alpha = \text{ord}(A, \leq)$$

$$\beta = \text{ord}(B, \leq_1)$$

$$\rho = \text{ord}(C, \leq_2) \tag{2}$$

and

$$A \cap B = \phi$$

$$A \cap C = \phi. \tag{3}$$

Since $\beta < \rho$, the set $(B, \leq_1)$ is order isomorphic to some segment of $(C, \leq_2)$, which we will denote by C_t. Hence

$$f : B \to C_t. \tag{4}$$

We form the sets $(A \cup B, \leq_1{}^*)$ and $(A \cup C, \leq_2{}^*)$ with the orders defined in Problem 8-14. Then

$$F : A \cup B \to A \cup C_t \tag{5}$$

where

$$F(x) = \begin{cases} x & \text{if } x \in A \\ f(x) & \text{if } x \in B \end{cases} \tag{6}$$

is an order isomorphism to $A \cup B$ onto a proper segment $A \cup C_t$ of $A \cup C$. Therefore,

$$\alpha + \beta < \alpha + \rho. \tag{7}$$

● PROBLEM 8-20

Prove that for any ordinal numbers α, β, and ρ

$$(\alpha + \beta = \alpha + \rho) \Rightarrow (\beta = \rho). \tag{1}$$

SOLUTION:

r.a.a. (reductio ad absurdum)

Suppose ordinal numbers exist, such that

$$\alpha + \beta = \alpha + \rho \quad \text{and} \quad \beta \neq \rho.$$

When $\beta \neq \rho$, then either $\beta < \rho$ or $\rho < \beta$.

Suppose $\beta < \rho$. From Problem 8-19, we have

$$(\beta < \rho) \Rightarrow (\alpha + \beta < \alpha + \rho), \tag{2}$$

which is a contradiction.

Rule (1) is called left cancellation. In Problem 8-21, we will show that right cancellation is not true for ordinal numbers.

● PROBLEM 8-21

Give a counter-example to show that the right cancellation rule is not true for ordinal numbers.

SOLUTION:

We have to find three ordinal numbers, such that

$$\beta + \alpha = \rho + \alpha \quad \text{and} \quad \beta \neq \rho.$$

In Problem 8-16, we proved

$$n + \omega = \omega. \tag{1}$$

Hence,

$$1 + \omega = \omega$$

and

$$2 + \omega = \omega. \tag{2}$$

We obtain

$$1 + \omega = 2 + \omega \quad \text{but} \quad 1 \neq 2.$$

● PROBLEM 8-22

Prove that for any ordinal numbers α, β, and ρ

$$(\beta < \rho) \Rightarrow (\beta + \alpha \leq \rho + \alpha). \tag{1}$$

SOLUTION:

Let $(A, \leq)$, $(B, \leq')$, and $(C, \leq'')$ represent well ordered sets, such that

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B, \leq'), \quad \rho = \text{ord}(C, \leq'') \tag{2}$$

and

$$B \cap A = \phi \quad C \cap A = \phi. \tag{3}$$

Since $\beta < \rho$, the set B is order isomorphic to some proper segment C' of C. Hence,

$$f : B \to C' \tag{4}$$

is an order isomorphism.

We shall use the reductio ad absurdum method.

Suppose that

$$\beta + \alpha > \rho + \alpha. \tag{5}$$

Hence, $C \cup A$ is order isomorphic to some proper segment $(B \cup A)'$ of $B \cup A$. Hence,

$$g : C \cup A \to (B \cup A)' \tag{6}$$

is order-isomorphic, which leads to contradiction.

● PROBLEM 8-23

Show that relation $\leq$ cannot be replaced by $<$ in Problem 8-22.

SOLUTION:

We have to find three ordinal numbers α, β, and ρ, such that

$$\beta < \rho \tag{1}$$

and

$$\beta + \alpha < \rho + \alpha. \tag{2}$$

We shall use the helpful formula,

$$k + \omega = \omega. \tag{3}$$

Let us substitute $\beta = 1$, $\rho = 2$, and $\alpha = \omega$ to obtain

$$1 < 2 \quad \text{and} \quad 1 + \omega = \omega = 2 + \omega. \tag{4}$$

Hence, (2) is not true.

• PROBLEM 8–24

Prove that for any ordinal number α, $\alpha + 1$ is an ordinal number and

$$\alpha < \alpha + 1. \tag{1}$$

SOLUTION:

It is easy to show that for any ordinal number α,

$$\alpha + 0 = 0 + \alpha = \alpha. \tag{2}$$

Substituting $\beta = 0$ and $\rho = 1$ in equation (1) of Problem 8-22, we obtain

$$(0 < 1) \Rightarrow (\alpha + 0 < \alpha + 1). \tag{3}$$

From (2) and (3), we conclude that for any ordinal number α,

$$\alpha < \alpha + 1. \tag{4}$$

• PROBLEM 8–25

Show that the greatest ordinal number does not exist.

SOLUTION:

Suppose, on the contrary, that the greatest ordinal number exists, and denote it by δ. From Problem 8-24, we know that for any ordinal number

$$\alpha < \alpha + 1.$$

Hence,

$$\delta < \delta + 1.$$

Therefore, $\delta + 1$ is an ordinal number which is larger than δ.

Contradiction. Conclusion: the greatest ordinal number does not exist.

• PROBLEM 8–26

Let α and β denote ordinal numbers, such that $\alpha \leq \beta$. Prove that a unique ordinal number ρ exists, such that

$$\alpha + \rho = \beta. \tag{1}$$

Number ρ is sometimes denoted by $(-\alpha) + \beta$.

SOLUTION:

First we shall show that if ρ exists, then it is unique.

Suppose, on the contrary, that ρ and ρ' exist, such that

$$\alpha + \rho = \beta \quad \text{and} \quad \alpha + \rho' = \beta. \tag{2}$$

Then

$$\alpha + \rho = \alpha + \rho' \tag{3}$$

and

$$\rho = \rho'. \tag{4}$$

If $\alpha = \beta$, then $\rho = 0$. Suppose $\alpha < \beta$; then sets A, B, B_t exist, such that $A \cap B = \phi$ and

$$\alpha = \operatorname{ord}(A, \leq), \quad \beta = \operatorname{ord}(B, \leq_1)$$

and an order isomorphism exists

$$f : A \to B_t$$

where B_t is a proper segment of B.

Let

$$\rho = \operatorname{ord}(B - B_t, \leq_1). \tag{5}$$

Then

$$\alpha + \rho = \beta \tag{6}$$

where $\alpha + \gamma = \operatorname{ord}(A \cup (B - B_t), \leq^*)$.

Order isomorphism can be defined as follows:

$$F(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in B - B_t \end{cases} \tag{7}$$

● PROBLEM 8–27

Prove that for any ordinal numbers α, β, and ρ

$$(\beta < \rho) \Leftrightarrow (\alpha + \beta < \alpha + \rho). \tag{1}$$

SOLUTION:

$\Rightarrow$ Demonstrated in Problem 8-19.

$\Leftarrow$ Since $\alpha + \beta < \alpha + \rho$, an ordinal number δ exists, such that (see Problem 8-26)

$$\alpha + \beta + \delta = \alpha + \rho. \tag{2}$$

By applying left cancellation, we obtain

$$\beta + \delta = \rho$$

which leads to

$$\beta < \rho \tag{3}$$

because $\delta \neq 0$.

● PROBLEM 8-28

Let

$$A = \{a, b, c, d\} \tag{1}$$

and

$$B = \{1, 2, 3\} \tag{2}$$

represent well ordered sets with orderings defined by:

for A $\quad a \leq b \leq c \leq d$

and for B $\quad 1 \leq 2 \leq 3.$

Find any well ordering of the Cartesian product of A and B.

SOLUTION:

Let us denote the sets by

$$(A, \leq) \text{ and } (B, \leq_1).$$

We have to determine when

$$(a, b) \leq^* (c, d) \tag{3}$$

where $a, c \in A$ and $(b, d) \in B$.

The elements of $A \times B$ can be well ordered as follows:

$$(a, 1) \leq^* (a, 2) \leq^* (a, 3) \leq^* (b, 1) \leq^* (b, 2) \leq^*$$

$$\leq^* (b, 3) \leq^* (c, 1) \leq^* (c, 2) \leq^* (c, 3) \leq^* (d, 1) \leq^*$$

$$\leq^* (d, 2) \leq^* (d, 3).$$

It is obvious that the set $(A \times B, \leq^*)$ is well ordered.

There are many ways (exactly 12!) of well ordering the set $A \times B$. We have just chosen one possibility.

● PROBLEM 8-29

Let $(A, \leq)$ and $(B, \leq_1)$ denote well ordered sets. Define the lexicographic (dictionary type) ordering of $A \times B$.

SOLUTION:

We construct the Cartesian product of the sets A and B. The lexicographic ordering is defined as follows:

$$(a,b) \leq {}^*(c,d) \text{ iff } \begin{cases} a < c \\ \text{or} \\ a = c \text{ and } b \leq_1 d \end{cases}$$

where $$a, c \in A \quad b, d \in B. \tag{1}$$

If both sets $(A, \leq)$ and $(B, \leq_1)$ are well ordered, then $(A \times B, \leq^*)$ is a well ordered set.

In a similar manner, we can impose the lexicographic ordering on the Cartesian product of n well ordered sets $(A_1, \leq_1)$, $(A_2, \leq_2)$, ..., $(A_n, \leq_n)$.

$$(x_1, x_2, \dots x_n) \leq {}^*(y_1, \dots, y_n) \text{ iff } \begin{cases} x_1 <_1 y_1 \\ x_1 = y_1 \text{ and } x_2 <_2 y_2 \\ x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } x_3 <_3 y_3 \\ \vdots \\ x_1 = y_1 \text{ and } \dots x_{n-1} = y_{n-1} \text{ and } x_n <_n y_n \end{cases}$$

where $x_1, y_1, \in A_1$, $x_2, y_2 \in A_2$, ..., $x_n, y_n \in A_n$.

● PROBLEM 8-30

Prove the following:

THEOREM

If $(A, \leq)$ and $(B, \leq_1)$ are well ordered sets, then the set $(A \times B, \leq^*)$ where $\leq^*$ is the lexicographic ordering is a well ordered set.

■

SOLUTION:

Since both sets A and B are totally ordered for any two pairs $(a,b) \in A \times B$ and $(c, d) \in A \times B$, we have either $(a, b) \leq^* (c, d)$ or $(c, d) \leq^* (a, b)$.

The set $(A \times B, \leq^*)$ is totally ordered.

To show that $(A \times B, \leq^*)$ is well ordered, we must show that every subset $\phi \neq T \subset A \times B$ contains the least element. Consider the following projection:

$$p_A(T) = \{x \in A : \text{for some } b \in B, (x, b) \in T\}. \tag{1}$$

The set $p_A(T)$ is a non-empty subset of a well ordered set $(A, \leq)$. Hence, it contains the least element; i.e., t_1. By the same token,

$$p_B(T) = \{y \in B : \text{for some } a \in A, (a, y) \in T\}$$

contains the least element; i.e., t_2. The element $(t_1, t_2) \in T$ is the least element of T.

Therefore, $(A \times B, \leq^*)$ is a well ordered set.

• PROBLEM 8–31

Prove the following:

THEOREM

Let $(A_1, \leq_1)$, $(A_2, \leq_2)$, $(B_1, \leq_1')$, and $(B_2, \leq_2')$ denote well ordered sets, such that

$$(A_1, \leq_1) \approx (A_2, \leq_2) \tag{1}$$

and

$$(B_1, \leq_1') \approx (B_2, \leq_2') \tag{2}$$

then

$$(A_1 \times B_1, \leq_1^*) \approx (A_2 \times B_2, \leq_2^*) \tag{3}$$

where $\leq_1^*$ and $\leq_2^*$ are lexicographic orderings.

■

SOLUTION:

Since $A_1 \approx A_2$ and $B_1 \approx B_2$, order isomorphisms exist, such that

$$f : A_1 \to A_2 \text{ and } g : B_1 \to B_2. \tag{4}$$

We will show that the function

$$F : A_1 \times B_1 \to A_2 \times B_2 \tag{5}$$

defined by

$$F(x, y) = (f(x), g(y)) \text{ for all } (x, y) \in A_1 \times B_1$$

is an order isomorphism.

Both functions f and g are bijections, thus F is a bijection. To complete the proof, we must show that

$$((a, b) \le_1{}^* (c, d)) \Rightarrow (F(a, b) \le_2{}^* F(c, d)) \tag{6}$$

where $(a, b), (c, d) \in A_1 \times B_1$.

If

$$(a, b) \le_1{}^* (c, d) \tag{7}$$

then two possibilities exist:

1. $a <_1 c$ and hence, f is an order isomorphism

$$f(a) <_2 f(\mathrm{c}). \tag{8}$$

Then

$$F(a, b) = (f(a), g(b)) \le_2{}^* (f(c), g(d)) = F(c, d). \tag{9}$$

2. $a = c$ and $b \le_1{}' d$. Functions f and g are order isomorphisms. Thus $f(a) = f(c)$ and $g(b) \le_2{}' g(d)$, and

$$F(a, b) = (f(a), g(b)) \le_2{}^* (f(a), g(d)) = F(c, d). \tag{10}$$

Thus, (6) is proven and F is an order isomorphism.

• PROBLEM 8–32

Define the ordinal product $\alpha\,\beta$ of ordinal numbers α and β, and explain why the ordinal product is not commutative.

SOLUTION:

DEFINITION

Let α and β denote ordinal numbers, the ordinal product $\alpha\,\beta$ is defined by

$$\alpha\,\beta = \mathrm{ord}(B \times A, \le^*) \tag{1}$$

where $(A, \le)$ and $(B, \le_1)$ are well ordered sets, such that

$$\alpha = \text{ord}(A, \le) \quad \beta = \text{ord}(B, \le_1)$$

and $\le^*$ is the lexicographic ordering of $B \times A$.

■

According to the definition above, the product $\alpha\,\beta$ is different from $\beta\,\alpha$. In general, the Cartesian product

$$A \times B \ne B \times A. \tag{2}$$

Furthermore, the lexicographic ordering of $A \times B$ is different from the lexicographic ordering of $B \times A$.

● PROBLEM 8–33

Show that

$$2\omega \ne \omega 2. \tag{1}$$

SOLUTION:

Let $(N, \le) = \{1, 2, 3, \dots\}$ and $(B, \le_1) = \{a, b\}$, so that

$$\text{ord}(N, \le) = \omega \quad \text{ord}(B, \le_1) = 2 \tag{2}$$

Hence,

$$2\omega = \text{ord}(N \times B, \le^*) \tag{3}$$

where $\le^*$ is the lexicographic ordering.

Let us define the function

$$f: N \times B \to N \tag{4}$$

$$f(n, a) = 2n - 1$$

$$f(n, b) = 2n.$$

It is easy to see that f is an order isomorphism. Thus

$$2\omega = \omega. \tag{5}$$

Now we shall form the set $(B \times N, \le^*)$ with the lexicographic ordering.

$$\text{ord}(B \times N, \le^*) = \omega 2 \tag{6}$$

$$(B \times N, \le^*) = \{(a, 1), (a, 2), \dots, (b, 1), (b, 2), \dots\} \tag{7}$$

We have

$$\text{ord}(B \times N, \leq^*) = \text{ord}((\{a\} \times N) \cup (\{b\} \times N), \leq^*) =$$

$$= \text{ord}(\{a\} \times N, \leq_1) + \text{ord}(\{b\} \times N, \leq_2) =$$

$$= \omega + \omega. \tag{8}$$

Thus,

$$2\omega \neq \omega 2. \tag{9}$$

● PROBLEM 8–34

Prove that for any finite ordinal k

$$k\omega = \omega. \tag{1}$$

SOLUTION:

Let $(A, \leq_1)$ denote a well ordered set

$$(A, \leq_1) = \{a_1, a_2, \ldots, a_k\}$$

and $(N, \leq_2) = \{1, 2, 3, \ldots\}$

$$\omega = \text{ord}(N, \leq_2) \tag{2}$$

Then

$$k\omega = \text{ord}(N \times A, \leq^*) \tag{3}$$

where $\leq^*$ is the lexicographic ordering. Consider the function

$$f : N \times A \to N \tag{4}$$

defined by

$$f(m, a_n) = k(m - 1) + n \tag{5}$$

where m is any natural number and $n = 1, 2, \ldots, k$.

Function f, defined by (4) and (5), is one-to-one and onto, hence it is a bijection.

Furthermore,

$$\left((n_1, a_{l_1}) \leq^* (n_2, a_{l_2})\right) \Rightarrow \left(f(n_1, a_{l_1}) \leq f(n_2, a_{l_2})\right). \tag{6}$$

Function f is order isomorphic. Hence,

$$k\omega = \omega. \tag{7}$$

Show that

1. $\alpha 0 = 0\alpha = 0$ for any ordinal number α.

2. $\alpha 1 = 1\alpha = \alpha$ for any ordinal number α.

3. $\alpha\beta = 0$, if and only if either $\alpha = 0$ or $\beta = 0$.

SOLUTION:

1. $\operatorname{ord}(A, \leq_1) = 0$, if and only if $A = \phi$. Let

$$\alpha = \operatorname{ord}(B, \leq_2). \tag{1}$$

Then

$$\alpha 0 = \operatorname{ord}(A \times B, \leq^*) = \operatorname{ord}(\phi \times B, \leq^*) =$$

$$= \operatorname{ord}(\phi, \leq^*) = 0. \tag{2}$$

Similarly,

$$\alpha 0 = \operatorname{ord}(B \times A, \leq_1{}^*) = \operatorname{ord}(\phi, \leq_1{}^*) = 0 \tag{3}$$

2. Let

$$1 = \operatorname{ord}(\{a\}, \leq_1) \quad \alpha = \operatorname{ord}(B, \leq_2). \tag{4}$$

Then

$$\alpha 1 = \operatorname{ord}(\{a\} \times B, \leq_1{}^*) = \operatorname{ord}(B, \leq_2) = \alpha. \tag{5}$$

Similarly,

$$1\alpha = \operatorname{ord}(B \times \{a\}, \leq_2{}^*) = \operatorname{ord}(B, \leq_2) = \alpha. \tag{6}$$

3. We shall prove that

$$(\alpha\beta = 0) \Leftrightarrow (\alpha = 0 \text{ or } \beta = 0). \tag{7}$$

$\Leftarrow$ was proven in part 1 of this problem.

$\Rightarrow$ Suppose $\alpha\beta = 0$. But

$$\operatorname{ord}(A, \leq) = 0, \text{ if and only if } A = \phi.$$

On the other hand,

$$\alpha\beta = \mathrm{ord}(B \times A, \leq^*). \tag{8}$$

Hence,

$$B \times A = \phi.$$

Therefore, either $A = \phi$ or $B = \phi$.

• PROBLEM 8–36

Prove the following:

THEOREM

Let α, β, and ρ denote ordinal numbers, such that $0 < \rho$. Then

$$(\alpha < \beta) \Rightarrow (\rho\alpha < \rho\beta). \tag{1}$$

■

SOLUTION:

Let $(A, \leq_1)$, $(B, \leq_2)$, and $(C, \leq_3)$ denote well ordered sets, such that

$$\alpha = \mathrm{ord}(A, \leq_1), \quad \beta = \mathrm{ord}(B, \leq_2), \quad \rho = \mathrm{ord}(C, \leq_3). \tag{2}$$

Since $\alpha < \beta$, $b \in B$ exists, such that

$$A \approx B_b. \tag{3}$$

We form the sets $(A \times C, \leq_1^*)$ and $(B \times C, \leq_2^*)$ with the lexicographic orderings. Hence,

$$(A \times C, \leq_1^*) \approx (B_b \times C, \leq_2') \tag{4}$$

where $\leq_2'$ is ordering inherited from $(B \times C, \leq_2^*)$. Let $c \in C$ denote the least element of C.

The set $B_b \times C$ is the segment of $B \times C$

$$B_b \times C = \{(x, y) \in B \times C : (x, y) \leq_2'' (b, c)\}. \tag{5}$$

Hence,

$$\rho\alpha < \rho\beta. \tag{6}$$

• PROBLEM 8–37

Show that, for any ordinal numbers α, β, and ρ, such that $0 < \rho$

$$(\rho\alpha = \rho\beta) \Rightarrow (\alpha = \beta). \tag{1}$$

SOLUTION:

Assume, on the contrary, that the ordinal numbers α, β, and ρ exist, such that $\rho > 0$ and $\rho\alpha = \rho\beta$ and $\alpha \neq \beta$.

Suppose $\alpha < \beta$.

Then by Problem 8-36, we obtain

$$\rho\alpha < \rho\beta \tag{2}$$

which is a contradiction.

It should be noted that ordinal multiplication is not right cancellative. The ordinal numbers α, β, and $0 < \rho$ exist, such that

$$\alpha\rho = \beta\rho, \quad \text{but} \quad \alpha \neq \beta.$$

For example,

$$2\omega = 1\omega \quad \text{but} \quad 1 \neq 2.$$

● PROBLEM 8-38

Prove the following important theorem.

THEOREM

Let α denote any ordinal number. Then the set

$$\{\beta : \beta \text{ is an ordinal, } \beta < \alpha\} \tag{1}$$

is a well ordered set whose ordinal number is α.

■

SOLUTION:

Let

$$\operatorname{ord}(A, \leq) = \alpha. \tag{2}$$

Then for any ordinal number $\beta < \alpha$ and any well ordered set $(B, \leq_1)$, such that

$$\beta = \operatorname{ord}(B, \leq_1) \tag{3}$$

the set $(B, \leq_1)$ is order isomorphic to a proper segment A_b, $b \in A$, of A. Element $b \in A$ is uniquely determined by the ordinal number β. This defines a function.

$$f : \{\beta : \beta < \alpha\} \to A \tag{4}$$

defined by

$$f(\beta) = b \tag{5}$$

where

$$\beta = \text{ord}(B, \leq_1) \text{ and } A_b \approx B. \tag{6}$$

Function f is a bijection. Furthermore,

$$(\beta_1 < \beta_2) \Rightarrow (f(\beta_1) \leq f(\beta_2)). \tag{7}$$

Hence, f is an order isomorphism and the set $\{\beta : \beta < \alpha\}$ is well ordered.

$$\text{ord}\{\beta : \beta < \alpha\} = \text{ord}\, A = \alpha. \tag{8}$$

● PROBLEM 8-39

Prove the following theorem:

THEOREM

Any set of ordinal numbers is well ordered.

■

SOLUTION:

r.a.a. (reductio ad absurdum)

Suppose a set of ordinal numbers A exists which is not well ordered. Then, at least one subset $B \subset A$ exists that does not have a least element. Hence, set B must contain a strictly decreasing infinite sequence of ordinal numbers

$$\beta_1 > \beta_2 > \beta_3 > \beta_4 \ldots$$

This sequence is contained in

$$\{\beta : \beta < \beta_1\}.$$

Hence, the set $\{\beta : \beta < \beta_1\}$ is not well ordered because it contains a strictly decreasing infinite sequence. That is a contradiction.

By theorem of Problem 8-38, this set is well ordered.

● PROBLEM 8-40

Use the theorem of Problem 8-38 to explain how each ordinal number α can be identified with the set

$$\{\beta : \beta < \alpha\}.$$

SOLUTION:

Let α denote an arbitrary ordinal number. Then the set

$$\{\beta : \beta < \alpha\} \tag{1}$$

is well ordered and

$$\alpha = \text{ord}\{\beta : \beta < \alpha\}. \tag{2}$$

Hence, we can identify the ordinal number α with the set $\{\beta : \beta < \alpha\}$. Each ordinal number can be regarded as a well ordered set of ordinal numbers. We have

$$\begin{aligned}
&0 \equiv \phi \\
&1 \equiv \{0\} \\
&2 \equiv \{0, 1\} \\
&3 \equiv \{0, 1, 2\} \\
&4 \equiv \{0, 1, 2, 3\} \\
&\quad\vdots \\
&\omega \equiv \{0, 1, 2, 3, \ldots\} \\
&\omega + 1 \equiv \{0, 1, 2, \ldots, \omega\} \\
&\omega + 2 \equiv \{0, 1, 2, \ldots, \omega, \omega + 1\} \\
&\omega 2 \equiv \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots\} \\
&\omega 2 + 1 \equiv \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega 2\} \\
&\quad\vdots
\end{aligned}$$

● PROBLEM 8-41

Russell's paradox leads to the conclusion that a set of all sets does not exist. Similar reasoning for the ordinal numbers is called the Burali-Forti paradox. Prove that:

THEOREM

A set of all ordinal numbers does not exist.

■

SOLUTION:

Suppose a set S of all ordinal numbers exists. By theorem of Problem 8-39, S is a well ordered set. Let s denote the ordinal number of S, then s

must be an element of S. We have

$$S = \text{ord}\,\{\alpha \in S : \alpha < s\} = \text{ord}\, S_s$$
$$< \text{ord}\, S \;\; = \text{s}$$

which is a contradiction.

CHAPTER 9

FUNDAMENTAL CONCEPTS OF TOPOLOGY

1. Which of the figures are congruent?

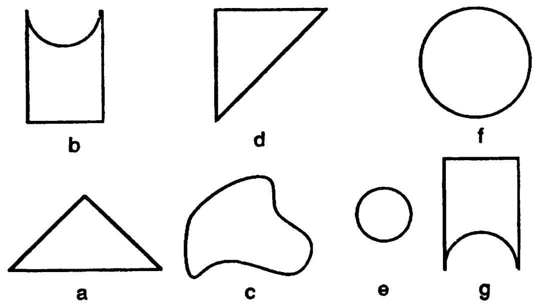


FIGURE 1

2. List a few properties of the triangle shown in Figure 2 which are geometric and a few which are not.

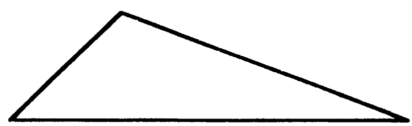


FIGURE 2

SOLUTION:

Geometry deals with figures and certain properties of figures in Euclidean space. To determine which properties are geometric and which are not, we shall introduce the notion of geometric equivalence, often called congruence.

Two figures are congruent if and only if one can be placed on top of the other, so as to fit perfectly.

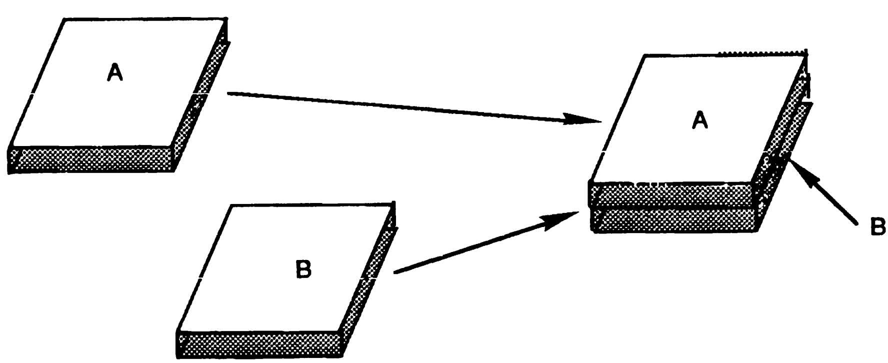


FIGURE 3

A and B shown in Figure 3 are congruent. The properties which are common for the congruent class of figures are called geometric properties.

1. From Figure 1, we see that a and d are congruent, b and g are congruent.

2. The geometric properties of the triangle are:
   the number of sides
   the length of its sides
   the number of angles
   the value of its angles
   the area enclosed by its perimeter.

The properties which are not geometric are:
   its color
   its orientation with respect to some given axes in the plane.

● PROBLEM 9-2

By using a rigid transformation (also called isometry), show that the triangles in Figure 1 are congruent.

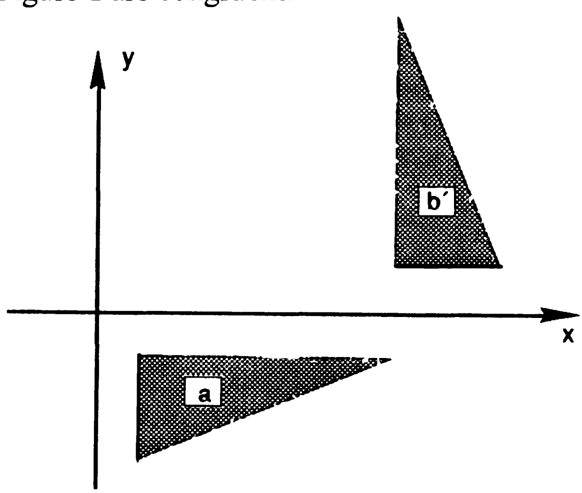


FIGURE 1

SOLUTION:

There are three fundamental rigid transformations: translation, rotation, and reflection. Figure 2 shows translation of a point P into P'.

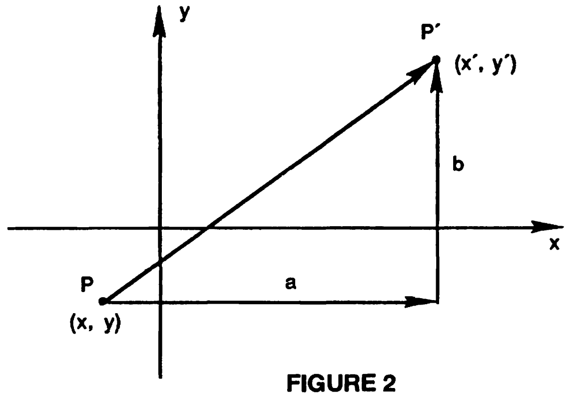


FIGURE 2

$$x' = x + a \qquad y' = y + b \tag{1}$$

or

$$(x', y') = (x, y) + (a, b) \tag{2}$$

Rotation of a point P about the origin is illustrated in Figure 3. The coordinates of P are (x, y).

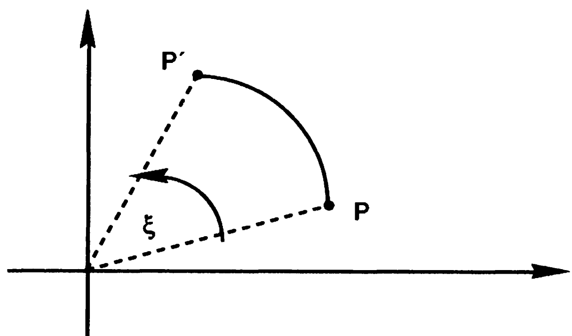


FIGURE 3

The coordinates of P' are (x', y') where

$$x' = x \cos \zeta - y \sin \zeta \quad y' = x \sin \zeta + y \cos \zeta \tag{3}$$

Suppose reflection takes place with respect to the x axis, then

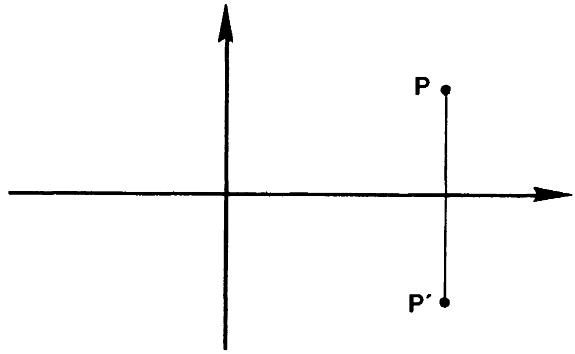


FIGURE 4

$$x' = x \qquad y' = -y$$

Now, back to the main problem (Figure 1).

Transformation yields a'. Rotation yields a'' and reflection yields a''', which is triangle b.

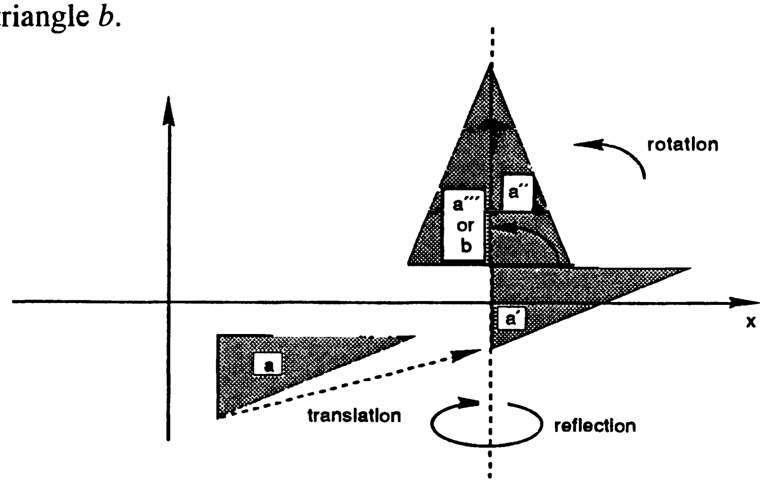


FIGURE 5

Thus, triangles a and b are congruent.

● PROBLEM 9–3

Why are rigid transformations isometric?

SOLUTION:

Isometric means preserving the distance.

Consider translation by a vector (a, b). Points A and B become A' and B'. The distance between A and B is

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

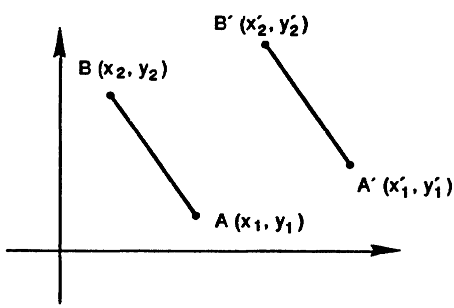


FIGURE 1

while the distance between A' and B' is

$$d_{A'B'} = \sqrt{(x_2'-x_1')^2 + (y_2'-y_1')^2} =$$

$$= \sqrt{[(x_2+a)-(x_1+a)]^2 + [(y_2+b)-(y_1+b)]^2} =$$

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = d_{AB} \tag{2}$$

Translation preserves the distances. Similarly, rotation preserves the distances. Consider rotation by angle ζ. From Problem 9-2, Equation (3), we obtain:

$$\sqrt{\begin{matrix}[(x_2\cos\zeta - y_2\sin\zeta) - (x_1\cos\zeta - y_1\sin\zeta)]^2 + \\ + [(x_2\sin\zeta - y_2\cos\zeta) - (x_1\sin\zeta - y_1\cos\zeta)]^2\end{matrix}} =$$

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}. \tag{3}$$

As for reflection, we can always assume that reflection takes place with respect to the x axis. Then

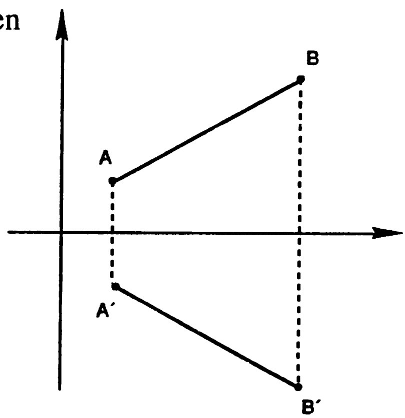


FIGURE 2

$$\sqrt{(x_2'-x_1')^2 + (y_2'-y_1')^2} = \sqrt{(x_2-x_1)^2 + ((-y_2)-(-y_1))^2} =$$

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \tag{4}$$

Conclusion: rigid transformations are isometric.

● PROBLEM 9-4

Show that the relation of congruence divides all geometric figures into disjoint classes. Explain the role that these classes play in Euclidean geometry.

SOLUTION:

Instead of writing

$$a \text{ is congruent to } b$$

we shall write

$$a \approx b. \tag{1}$$

Note that since

$$a \approx a \tag{2}$$

each geometric figure belongs to some class. Furthermore, if

$$a \approx b \text{ and } b \approx c, \text{ then } a \approx c. \tag{3}$$

Also, if $a \approx b$, then $b \approx a$.

Congruence is an equivalence relation. It separates the set of figures in Euclidean space into disjoint equivalence classes. Geometry deals with the equivalence classes and not with the particular elements of some class. Within each equivalence (congruence) class, all elements share the same geometric properties.

Suppose we are given two figures a and b. If we can point out one geometric property which these figures do not have in common, then a and b belong to two different congruence classes.

It is quite possible that these figures share many other geometric properties.

● PROBLEM 9-5

Suppose the congruence classes are given as described in Problem 9-1. Can you give an example (or two) of how to subdivide further each class?

SOLUTION:

We can take into account the location of figures in the space. For example, a class of squares of a certain size can be subdivided according to the

distance of their centers from the origin of the coordinate system. All the squares (of the same size) whose centers are located at distance p from the origin, belong to the same subclass as shown in Figure 1.

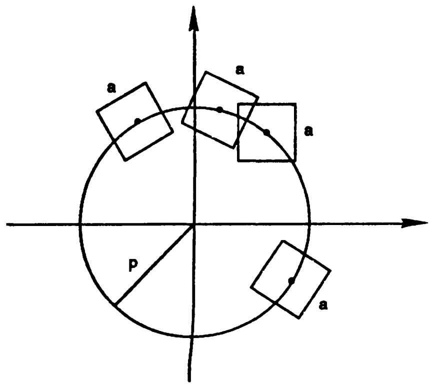


FIGURE 1

Here is another example. Suppose a line is given. The figures of some congruence class belong to the same subclass when their sides make the same angles with the line (see Figure 2).

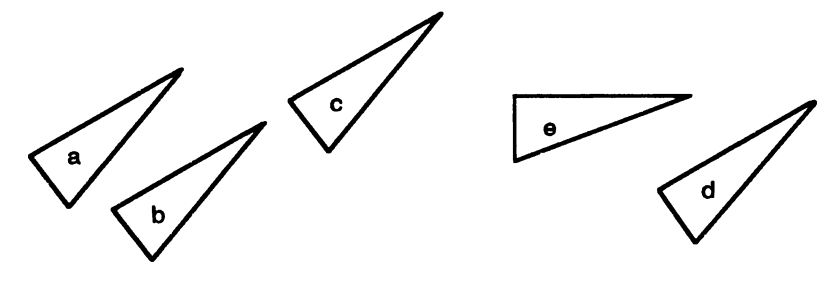


FIGURE 2

Note that triangles *a, b, c,* and *d* belong to the same subclass, while *e* does not. Often, vectors are subdivided into the classes in this manner; that is, vectors with equal length and orientation belong to the same class.

● PROBLEM 9–6

Remember the geometry described in Problem 9-4. We shall move a step further and define what might be called geometry of the magnifying glass or similarity geometry. Now, within each equivalence class, we permit right transformations and proportional magnification or contraction.

1. What are the conditions for two rectangles to belong to the same equivalence class?

2. Name a few geometric properties which remain invariant under the permitted transformations.

SOLUTION:

1. Note that all straight line segments belong to the same equivalence class in similarity geometry. Also, all squares are equivalent, and all circles are equivalent. Obviously, area is no longer an invariant.

The rectangles with the same ratio of side lengths are equivalent.

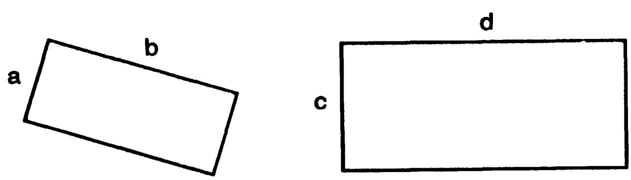


FIGURE 1

Rectangles of Figure 1 are equivalent, if and only if $b/a = d/c$.

2. The values of angles remain the same under the new transformations. Also, straight line segments continue to be straight line segments. In similarity geometry, the overall shape of the figures is preserved. Certain congruence classes of ordinary geometry have been combined to form equivalence classes of similarity geometry (see Figure 2).

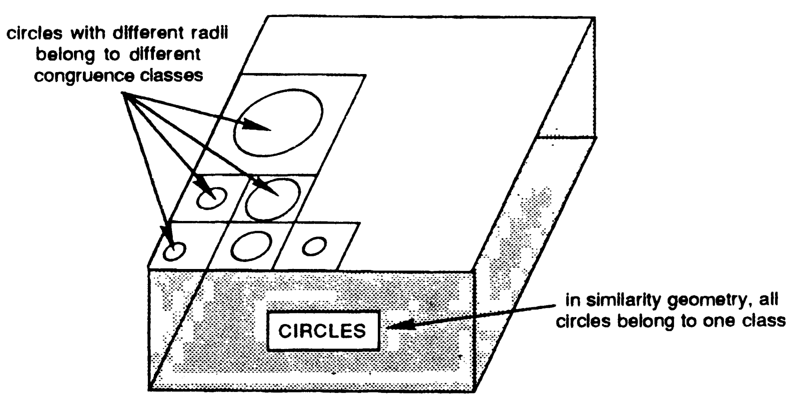


FIGURE 2

As a next step on our way to more and more general equivalence classes, we shall define affine geometry. The permitted transformations in the plane are defined now by

$$(x, y) \rightarrow (x', y') \tag{1}$$

where

$$x' = ax + by + c \quad y' = dx + ey + f. \tag{2}$$

The numbers a, b, c, d, e, and f are real and, such that

$$ae - bd \neq 0. \tag{3}$$

Show that in affine geometry the lines which were originally parallel, remain parallel although the angles are not invariant.

SOLUTION:

Suppose two parallel lines AB and CD are given, as shown in Figure 1 below:

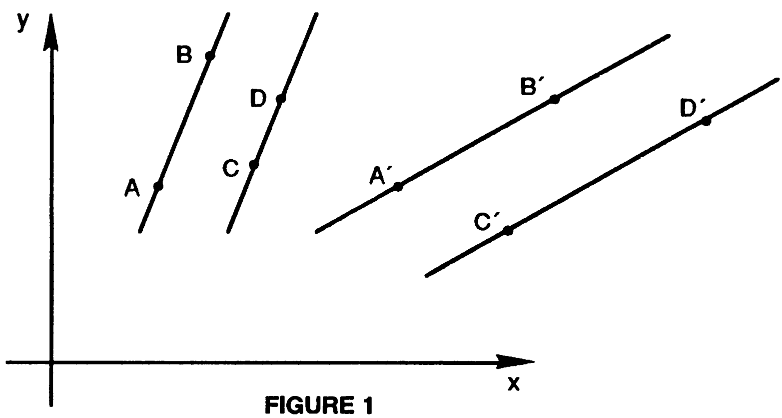


FIGURE 1

We denote the coordinates of the points A, B, C, and D by (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). Since the lines are parallel, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}. \tag{4}$$

Under transformation (2), the points A, B, C, D are transformed to A', B', C', and D' to correspond with coordinates

$$A' \ (ax_1 + by_1 + c, dx_1 + ey_1 + f) = (x_1', y_1')$$

$$B' \ (ax_2 + by_2 + c, dx_2 + ey_2 + f) = (x_2', y_2')$$

$$C' \quad (ax_3 + by_3 + c, dx_3 + ey_3 + f) = (x_3', y_3')$$

$$D' \quad (ax_4 + by_4 + c, dx_4 + ey_4 + f) = (x_4', y_4'). \tag{5}$$

To show that the lines $A'B'$ and $C'D'$ are parallel, we must prove that

$$\frac{y_2' - y_1'}{x_2' - x_1'} = \frac{y_4' - y_3'}{x_4' - x_3'}. \tag{6}$$

Indeed,

$$\frac{d(x_2 - x_1) + e(y_2 - y_1)}{a(x_2 - x_1) + b(y_2 - y_1)} = \frac{d + e\dfrac{y_2 - y_1}{x_2 - x_1}}{a + b\dfrac{y_2 - y_1}{x_2 - x_1}} =$$

$$= \frac{d + e\dfrac{y_4 - y_3}{x_4 - x_3}}{a + b\dfrac{y_4 - y_3}{x_4 - x_3}} = \frac{d(x_4 - x_3) + e(y_4 - y_3)}{a(x_4 - x_3) + b(y_4 - y_3)}. \tag{7}$$

Hence, lines $A'B'$ and $C'D'$ are parallel.

● PROBLEM 9–8

1. Using drawings, explain shear and strain.

2. Show that all triangles are equivalent in affine geometry and not all four-sided polygons are equivalent.

SOLUTION:

1. Shear is illustrated in Figure 1.

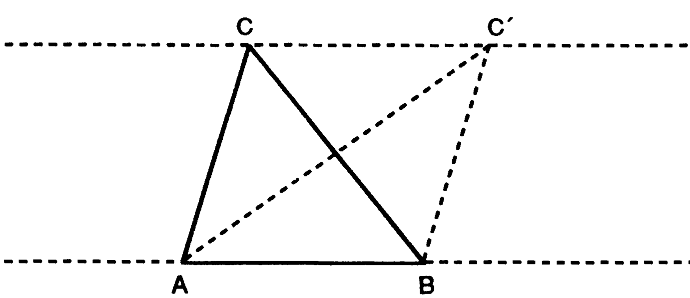


FIGURE 1

Here, the base of the triangles remains the same, while the vertex C is moved

along a line parallel to the base.

Another transformation allowed in affine geometry is strain. Here the vertex is moved along a line which is not parallel to the base. See Figure 2.

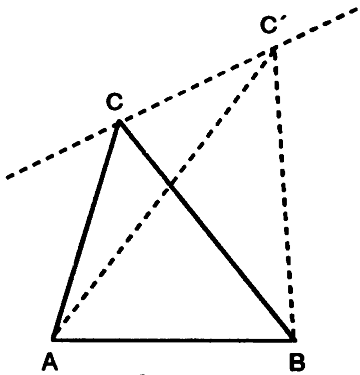


FIGURE 2

2. Suppose we are given any two triangles. Then by using only affine transformations, we can transform one triangle into another. Hence, all triangles belong to one equivalence class in affine geometry. Consider the two four-sided polygons, shown in Figure 3.

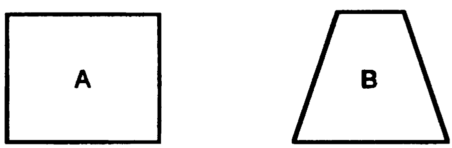


FIGURE 3

Since affine transformations transform parallel lines into parallel lines, A cannot be transformed into B. Hence, A and B belong to two different equivalence classes in affine geometry.

● PROBLEM 9-9

Explain why the equivalence classes of affine geometry are combinations of equivalence classes of similarity geometry which are, in fact, combinations of equivalence (congruence) classes of ordinary geometry.

SOLUTION:

Each geometry class is determined by the set of transformations it allows.

Ordinary Geometry ↓ Rigid Transformations	Similarity Geometry ↓ Rigid Transformations and Contraction and Magnification	Affine Geometry ↓ Rigid Transformations (Shear, Strain)

FIGURE 1

Each transformation allowed in ordinary geometry is allowed in similarity geometry, and each transformation allowed in similarity geometry is also allowed in affine geometry. The situation is illustrated in Figure 2.

Equivalence classes of

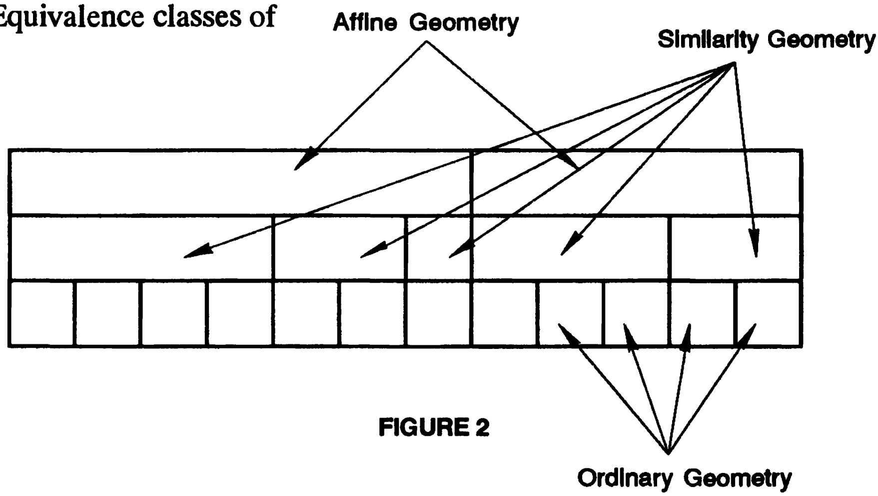


FIGURE 2

• PROBLEM 9–10

> One of the invariants of affine geometry is that the parallel lines are transformed into parallel lines. Show that this is not true for projective geometry.

SOLUTION:

In projective geometry, the projective transformations are permitted. Such transformations are perspective projections of a figure.

In Figure 1, the triangle ABC in one plane is transformed into $A'B'C'$ in the other plane by projection from an exterior point. We shall also consider parallel projections, as shown in Figure 2.

Obviously, two parallel lines can be projected into two lines which are not parallel.

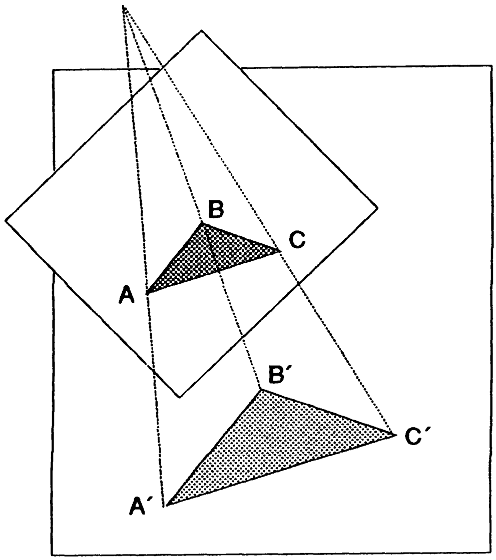


FIGURE 1

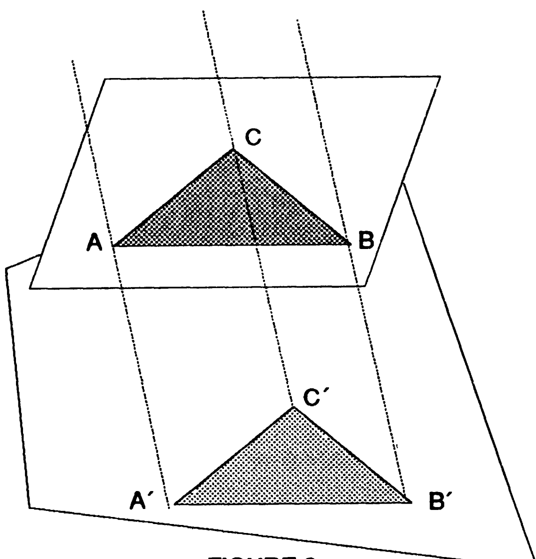


FIGURE 2

● PROBLEM 9-11

In projective geometry, straight lines remain straight lines. Show that cross-ratio is also an invariant; that is, prove that

$$\frac{AC:BC}{AD:BD} = \frac{A'C':B'C'}{A'D':B'D'}. \tag{1}$$

SOLUTION:

Here we shall prove (1) for projection from an exterior point. The proof

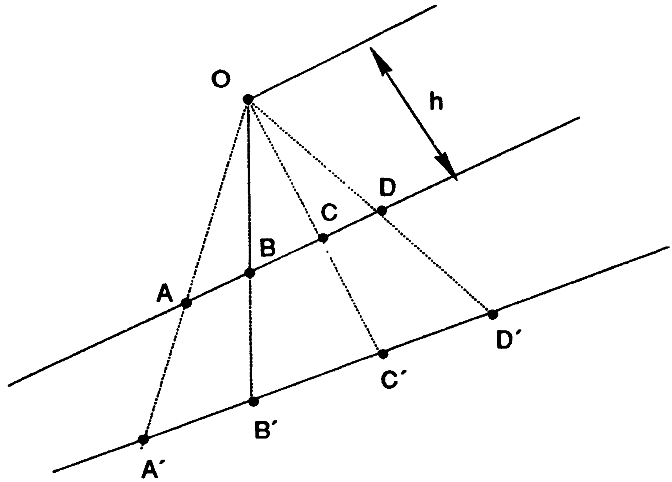


FIGURE 1

for the parallel projection is similar. Consider the triangle AOC. Its area is equal to

$$S_{AOC} = {}^1\!/_2\, h{\cdot}AC = AO{\cdot}OC \sin AOC. \tag{2}$$

We obtain similar equations for all involved triangles. Thus,

$$\frac{AC{:}BC}{AD{:}BD} = \frac{S_{AOC}{:}S_{BOC}}{S_{AOD}{:}S_{BOD}} =$$

$$= \frac{AO\cdot OC\cdot \sin AOC}{BO\cdot OC\cdot \sin BOC}\cdot\frac{BO\cdot OD\cdot \sin BOD}{AO\cdot OD\cdot \sin AOD} =$$

$$= \frac{\sin AOC\cdot \sin BOD}{\sin BOC\cdot \sin AOD}. \tag{3}$$

The final result in (3) does not depend on the choice of points A', B', C', and D'; hence, cross-ratio is an invariant.

● PROBLEM 9–12

All the geometries considered, so far, have had one important invariant in common: a straight line remained a straight line. This is not true in topology, where a square can be transformed into a circle, since we allow all one-to-one bi-continuous transformations. The new transformations, now permitted, are called elastic deformations.

Bending, stretching, and twisting is now allowed. Cutting is not allowed, unless the cuts are repaired in such a way that the points which were "near" remain "near."

Which figures shown in Figure 2 and Figure 3 belong to the same topological equivalence classes?

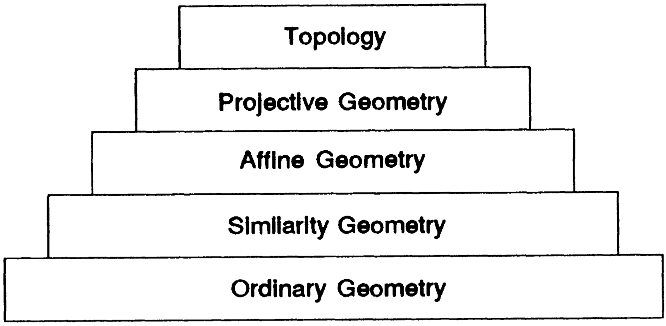


FIGURE 1

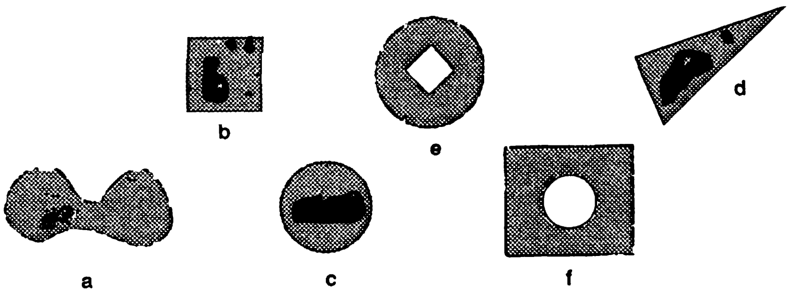


FIGURE 2

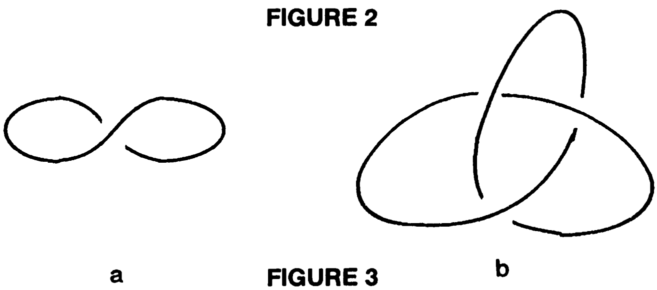


FIGURE 3

SOLUTION:

Figures *a, b, c,* and *d* of Figure 2 belong to the same topological equivalence class. Each may be transformed into any of the others by elastic deformations.

Figures *e* and *f* belong to the same class.

Figure 3 shows two plane curves which belong to the same topological class. In three-dimensional space, *a* cannot be transformed into *b* without cutting and subsequent re-connecting.

● PROBLEM 9–13

The plane closed curve C is deformed into C^{I}, then into C^{II}, then into C^{III} and finally into C^{IV}, see Figure 1. Which of the transformations are

topological?

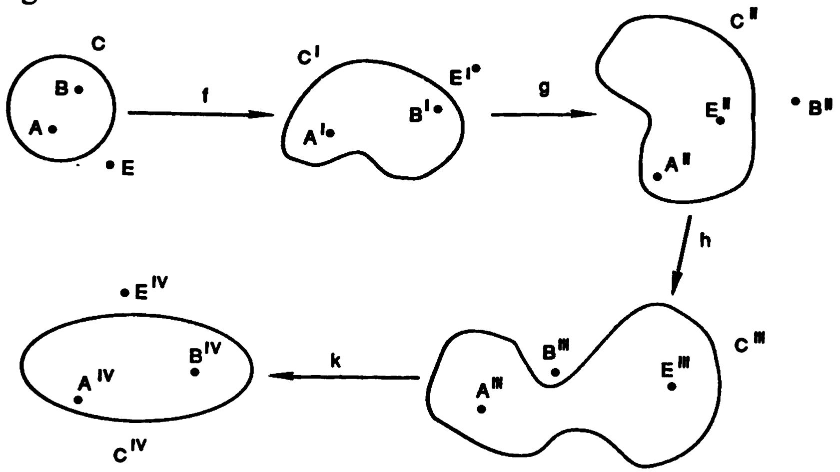


FIGURE 1

SOLUTION:

In topology the distances are much less important than in the geometries defined so far. Here we investigate non-metric spatial relationships. What is important is the preservation of the "nearness" of the points. We understand "nearness" in the topological sense. The property of separating a plane surface into a region outside and a region inside the plane closed curve is a topological invariant.

We conclude that f(of Figure 1) is a topological transformation. Points A and B inside C are transformed into A^I and B^I inside C^1 while the point E outside C is transformed into E^I outside C^I. Transformations g and k are not topological while h is a topological transformation.

● PROBLEM 9–14

Consider the curves C_1 and C_2.

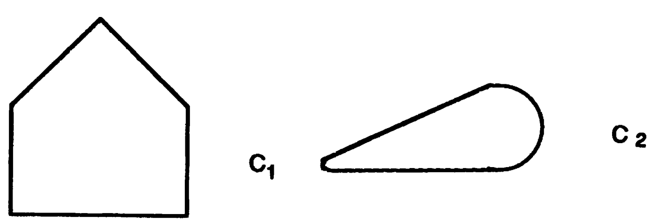


FIGURE 1

Which of the following properties are topological, which are geometric, and which are neither?

1. C_1 is above C_2.

2. The area enclosed by C_1 is larger than the area enclosed by C_2.

3. C_1 and C_2 have no common points.

4. C_1 consists of straight lines, while C_2 consists of straight lines and a semicircle.

5. C_1 has more vertices than C_2.

SOLUTION:

1. This property is neither geometric nor topological. Rotation can place C_2 above C_1.

2. The area is an invariant in ordinary geometry. But under affine transformations, the area is not preserved. This is not a topological property.

3. This is a geometric and topological property.

4. This is not a topological property. It is a geometric property in ordinary geometry.

5. It is a geometric but not topological property.

● PROBLEM 9–15

There is a sphere with a continuous closed non-self-intersecting curve C drawn on its surface. The curve C divides the surface into two disjoint regions A_1 and A_2. One cannot move from any point in A_1 to any point in A_2 without crossing C, see Figure 1.

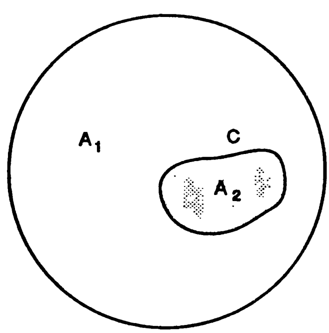


FIGURE 1

1. How many times does a path which joins a point belonging to one region with a point in the other region cross the curve *C*? What happens when both points belong to the same region?

2. Is it possible, by continuous transformations, to contract *C* to a point?

SOLUTION:

1. Figure 2 depicts a sphere with a curve *C*.

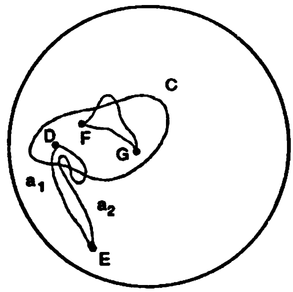


FIGURE 2

Points *D* and *E* belong to different regions; path *a*, crosses *C* once, a_2 three times. Hence, a point belonging to one region may be joined to a point belonging to the other by paths which cross *C* an odd number of times.

When both points belong to the same region (points *F* and *G*), the path from *F* to *G* will cross *C* an even number of times.

2. Curve *C*, which is closed, non-self-intersecting, and continuous may gradually be contracted on the surface of a sphere to a point. This is an important property, which we shall discuss later in detail.

● PROBLEM 9–16

The surface of a one-fold torus is shown in Figure 1. It has the form of a well-known doughnut. (If you are on a diet, compare it to a bicycle tire).

1. Curve *C* separates the surface into two disjoint regions, and it may be continuously contracted on the surface into a point. Is this true for any continuous non-self-intersecting closed curve?

2. What is the maximum number of continuous non-self-intersecting closed curves which may be drawn on the surface of a torus without dividing it?

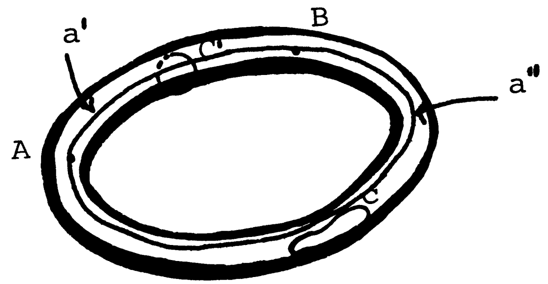


FIGURE 1

SOLUTION:

1. Consider curve C' of Figure 1. It cannot be continuously contracted into a single point. Furthermore, C' does not divide the surface of the torus into two regions. Points A and B can be joined by path a'' which does not cross C'.

The sphere and the torus belong to different topological equivalence classes. Topologically equivalent surfaces are called homeomorphic.

2. One. If we draw any other closed curve besides C (as in Figure 2), the surface of the torus will be divided. Any path AB crosses either C or C' at least once.′

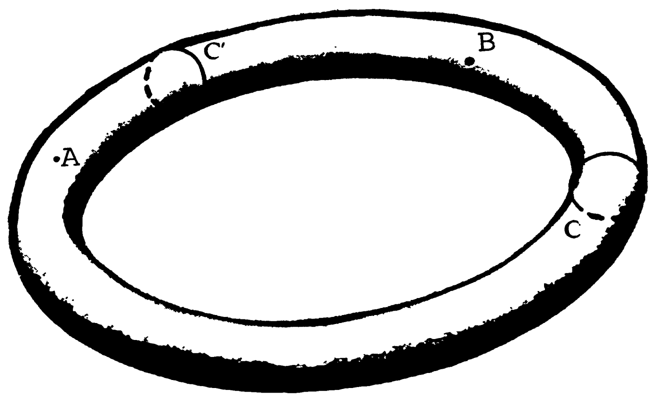


FIGURE 2

● PROBLEM 9–17

We shall move a step further and investigate the curves on the surface of a two-fold torus, see Figure 1.

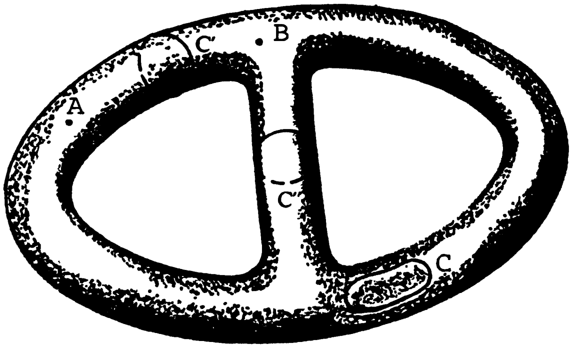


FIGURE 1

How many continuous, closed curves can be drawn on the surface of this torus without dividing it into distinct regions?

SOLUTION:

Curve C, which can be continuously contracted into a point, divides the torus into two regions. Hence, we shall investigate only the curves which cannot be contracted into a point. Draw curves C' and C''. Any two points A and B can be joined without crossing C' or C''. Adding one more curve C''' (see Figure 2) changes the situation. Any path AB has to cross at least one of the curves at least once. The maximum number of curves which do not divide a two-fold torus is two.

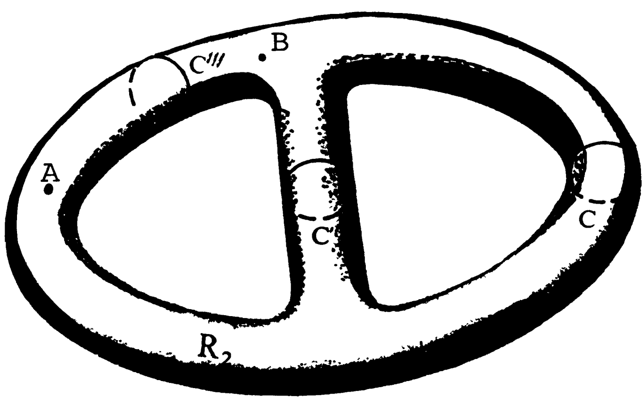


FIGURE 2

What is the genus of the surface of:

1. a sphere
2. a one-fold torus
3. an n-fold torus
4. a fork with a hot-dog on it
5. a ladder with five rungs.

SOLUTION:

The greatest number of continuous non-self-intersecting closed curves which may be drawn on a surface without dividing it into distinct regions defines the genus of the surface.

1. The genus of the surface of a sphere is 0.
2. The genus of the surface of a one-fold torus is 1.
3. Generally, the genus of an n-fold torus is n.
4. The genus of a fork with a hot dog on it is three, as shown in Figure 1.

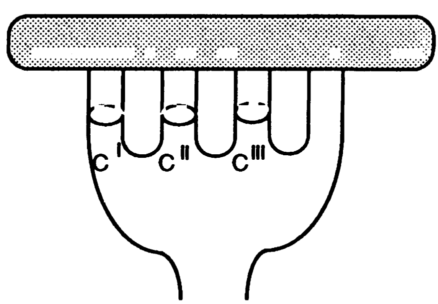


FIGURE 1

5. A ladder with five rungs has a genus of 4.

The genus of a surface is invariant under all one-to-one bicontinuous transformations; hence, it is a topological property. All surfaces belonging to the same topological equivalence class have the same genus.

● PROBLEM 9–19

Which of the following surfaces are closed and which are open?

1. a sphere
2. an n-fold torus
3. a hollow cylinder
4. a cube.

SOLUTION:

A surface is closed if it has no boundary curves. Otherwise, the surface is called open. A sphere, an n-fold torus, and a cube are closed surfaces.

The edges of a cube are not boundary curves. A hollow cylinder is an open surface. Similarly, a disc is open. Boundary curves of two-sided surfaces are curves which separate one side of a surface from the other.

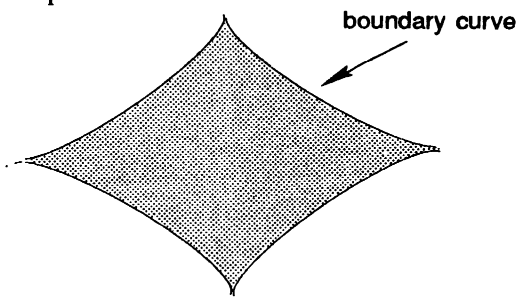


FIGURE 1

The edges of a piece of infinitely thin paper are boundary curves of this surface.

● PROBLEM 9–20

Describe the Möbius band, which is a one-sided surface. What is the boundary curve of the Möbius band? (See following figure.)

SOLUTION:

To obtain the Möbius band, take a rectangular strip of paper $ABCD$, make a 180° twist with the end DC and join up the ends. The obtained surface does not have two sides; it cannot be painted in different colors. An ant walking

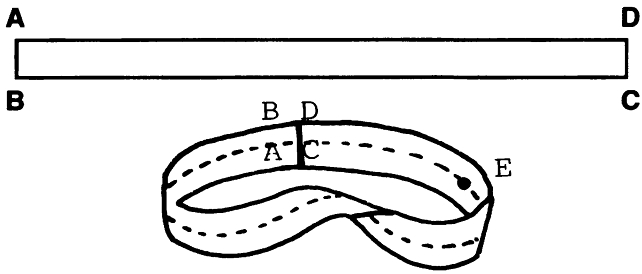


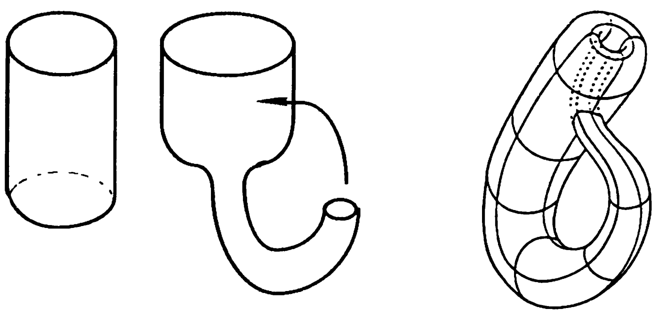
FIGURE 1

along a dotted line from a point E after making one round will end up on the other side of E without crossing the boundary line.

The Möbius band has only one boundary curve, which is topologically equivalent to a circle.

● PROBLEM 9–21

Why is the surface of a Klein bottle closed and one-sided?



a b c

FIGURE 1

SOLUTION:

We start with an open cylinder, stretch out and bend one end, then pass through the side surface without intersecting or breaking it. At the final stage, we join up the ends.

This bottle cannot be physically constructed in the three-dimensional space. Similarly, the Möbius band, which is a two-dimensional structure, cannot be constructed in two dimensions, because of the twist. Any two points on the surface of a Klein bottle may be joined by a continuous path not crossing any boundary curve. The surface of a Klein bottle has no bound-

ary curve. Note that two open ends of the original cylinder were joined together in such a way that the outside of one was connected to the inside of the other. Hence, the outside and inside became impossible to distinguish.

The surface of a Klein bottle is one-sided and closed.

● PROBLEM 9–22

Why is a Möbius band a non-orientable surface?

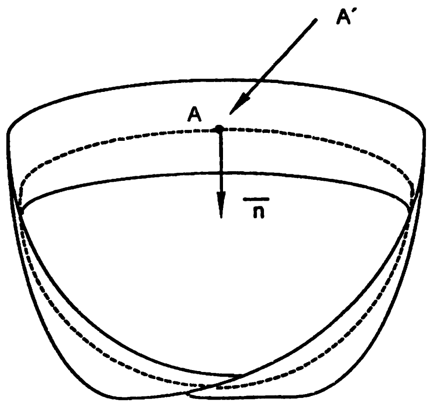


FIGURE 1

SOLUTION:

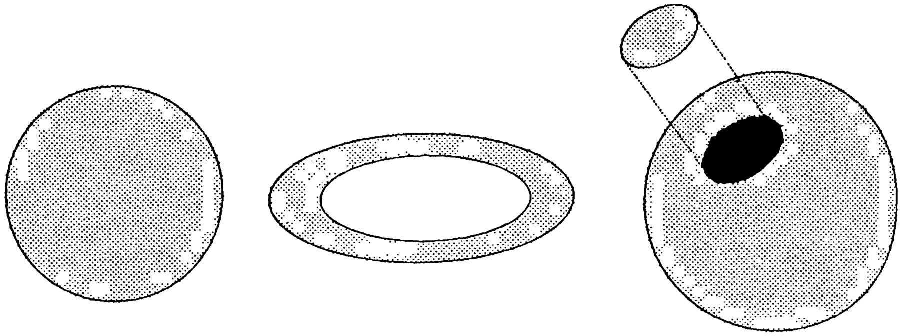
Consider a point A on the surface of a Möbius band. Let $\bar{n}$ be a normal to the surface at A. Suppose $\bar{n}$ moves along a dotted line from A to A'. The normal of A'has the opposite direction of that at A. Note that $\bar{n}$ is continuously defined as it moves from A to A'. Similarly, the surface of a Klein bottle is nonorientable.

● PROBLEM 9–23

A one-sided surface is connected if it is possible to join any two points of this surface with a continuous path. A two-sided surface is connected if its sides, taken separately, are both connected. Which of the surfaces depicted in Figure 1 is connected?

SOLUTION:

It is important to keep in mind the distinction between a solid and the surface of a solid. In most cases of a sphere, a torus, etc., we understand a surface is not a solid.



a b c

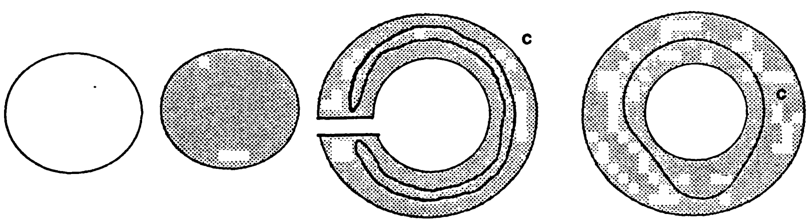
FIGURE 1

Hence, a sphere and a torus (Figure 1) are two-sided connected surfaces.

In Figure 1 *c* a disc is separated from a sphere. The total surface is disconnected. Now the total surface consists of two surfaces (each connected in itself) which are homeomorphic to a disc.

● PROBLEM 9-24

Which of these surfaces are simply connected?



a b c d

FIGURE 1

SOLUTION:

A surface is said to be simply connected if every continuous non-self-intersecting closed curve drawn on it may be continuously contracted on the surface into a point. The disc (a) and the sphere (b) are simply connected.

The surface shown in (c) is topologically equivalent to (a); hence, it is simply connected. The annulus (d) is not simply connected. Curve *C* cannot be continuously contracted to a point.

Annulus is doubly connected, since it requires only one cut to make it homeomorphic to a disc.

Which of the surfaces is quadruply connected?

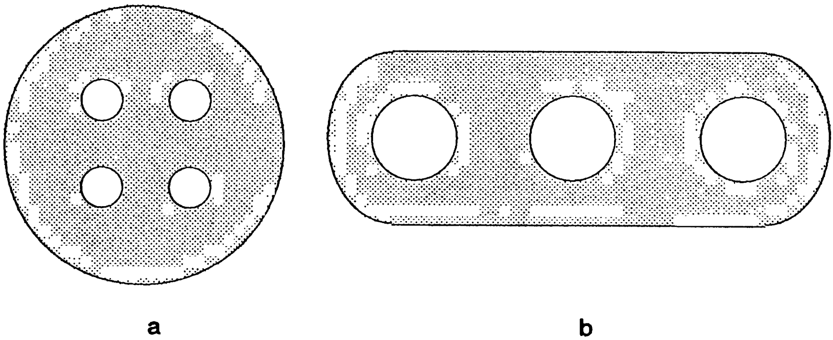


FIGURE 1

SOLUTION:

The surface is n-tuply connected if it requires $n - 1$ cuts in order to make it homeomorphic to a disc. Thus, annulus is doubly connected; it requires one cut to render it homeomorphic to a disc. Surface (a) of Figure 1 requires four cuts; hence, it is quintuply connected ($n = 5$).

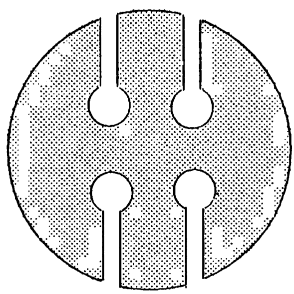


FIGURE 2

Surface (b) of Figure 1 is quadruply connected. It requires three cuts which can be made in many different ways (See Figure 3 (a) and (b)).

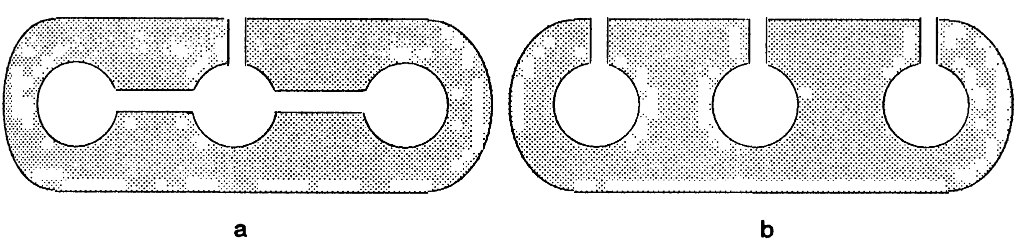


FIGURE 3

Note that each new cut forms a boundary of the surface, and the cuts cannot intersect.

● PROBLEM 9-26

Determine n for these n-tuply connected surfaces:

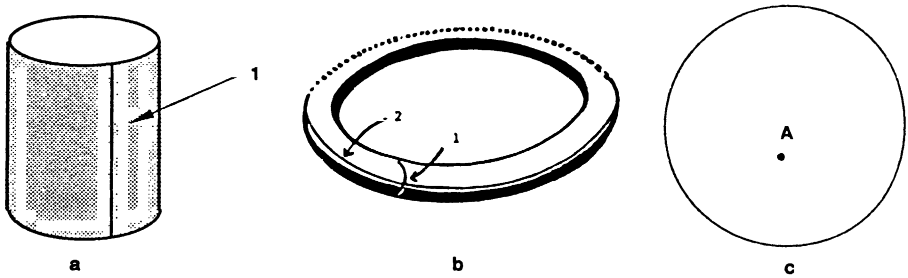


FIGURE 1

where (a) is the curved surface of a cylinder, (b) is the torus, and (c) is the sphere with one point removed.

SOLUTION:

We can cut the cylinder along line 1 and obtain a rectangle, which is homeomorphic to a disc; hence, $n = 2$.

We can cut the torus along line 1 to obtain a cylinder and then along 2. Thus, $n = 3$ and the surface of the torus is triply connected.

A sphere with a single hole is homeomorphic to a disc, $n = 1$.

● PROBLEM 9-27

Show that any two continuous non-self-intersecting closed curves on the sphere are homotopic. (Figures shown on following page.)

How many homotopy classes does the torus have?

SOLUTION:

Any continuous closed non-self-intersecting figure on a simply connected surface can be continuously contracted to a point. It follows that, on a simply connected surface, any curve may be continuously deformed into any other curve. C in Figure 1 can be continuously deformed into C'. All continuous

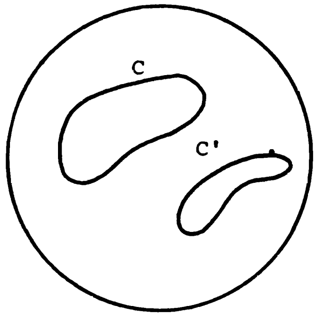


FIGURE 1

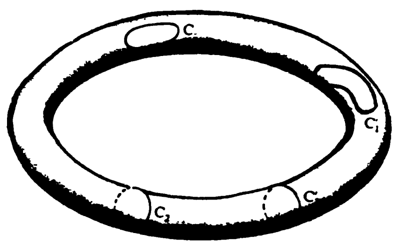


FIGURE 2

closed non-self-intersecting curves on a simply connected surface are homotopic to each other.

On the torus, C_1 can be transformed to C_1' and C_2 can be transformed to C_2'. But it is impossible to continuously transform C_1 into C_2. Curves C_1 and C_1' are homotopic, and curves C_2 and C_2' are homotopic. There are two homotopy classes for the surface of the torus.

Curves like C_1 and C_1', which may be continuously contracted to a point, belong to the null homotopy class.

● PROBLEM 9–28

What is the rank of a two-fold torus with a removed disc?

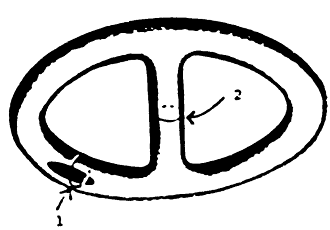


FIGURE 1

SOLUTION:

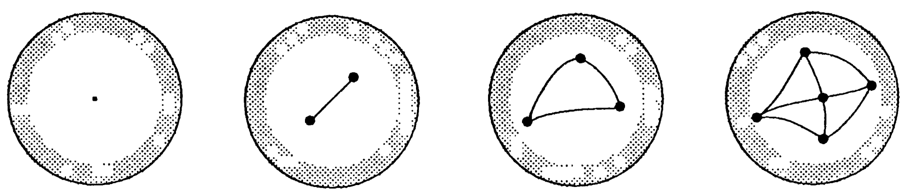
Let us start with a definition: The rank of an open surface is the least number of cuts required to make the surface homeomorphic to a disc. The rank of an n-tuply connected surface is $n - 1$.

The rank of a closed surface is the rank of the open surface obtained from the closed surface by the removal of a single disc. A disc and a sphere have rank zero. An annulus, the curved surface of a cylinder, and a Möbius band have rank 1.

Three cuts are required to make the surface shown in the figure homeomorphic to a disc.

We can also use an alternative definition of rank. Rank is the greatest number of non-intersecting cuts which can be made without making the surface disconnected.

● PROBLEM 9-29



a b c d

FIGURE 1

Verify the Euler rule for the figures *a, b, c*, and *d* drawn on a sphere.

SOLUTION:

Let us draw a figure having the number of vertices V of regions (or faces) F and of edges (or arcs) E, on a sphere.

Then, any figure drawn on a surface of a sphere satisfies the following rule:

$$V + F - E = 2 \tag{1}$$

discovered by Euler.

For a, we have:

$$V = 1, \quad F = 1, \quad E = 0 \tag{2}$$

hence $1 + 1 - 0 = 2$.

For b, we have

$$V = 2, \quad F = 1, \quad E = 1 \tag{3}$$

hence $2 + 1 - 1 = 2$.

For c, we have

$$V = 3, \quad F = 2, \quad E = 3 \tag{4}$$

hence $3 + 2 - 3 = 2$.

For d, we have

$$V = 5, \quad F = 5, \quad E = 8 \tag{5}$$

hence $5 + 5 - 8 = 2$.

In the next problem, we shall prove Euler's formula.

• PROBLEM 9-30

Prove Euler's formula:

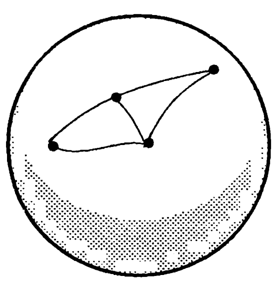
$$V + F - E = 2 \tag{1}$$

for the figures drawn on a sphere.

SOLUTION:

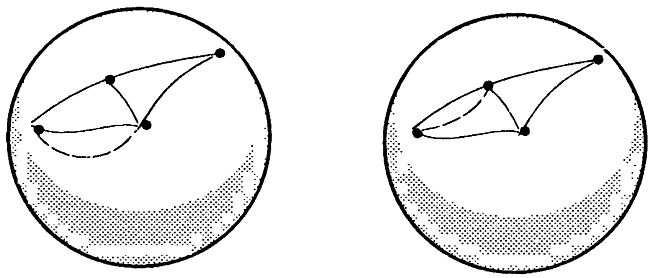
Suppose (1) is true for some figure depicted in Figure 1.

Let us draw a line connecting a vertex to another vertex (which does not cross any existing edges), as shown in Figure 2 a and b:



a

FIGURE 1



a b

FIGURE 2

Let Δ denote the increase, then

$$\Delta V = 0, \quad \Delta F = 1, \quad \Delta E = 1. \tag{2}$$

Then

$$V + \Delta V + F + \Delta F - E - \Delta E = V + F - E = 2 \tag{3}$$

and (1) holds.

Suppose we draw a line which ends with a new vertex (Figure 3).

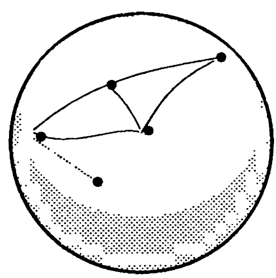


FIGURE 3

Then, $\Delta V = 1$, $\Delta F = 0$, $\Delta E = 1$ and (1) holds again.

Any figure on the sphere can be drawn, beginning with a vertex and then adding lines one after another. Hence, Euler's rule is proved.

● PROBLEM 9–31

> Find one or two simple regular divisions of a sphere. A division is called regular when every face has the same number of lines f as a boundary and the same number of lines v meet at every vertex.

SOLUTION:

Euler's formula states that

$$V + F - E = 2. \tag{1}$$

Since there are F faces on the sphere, the total number of lines (edges) is

$$\frac{Ff}{2} = E \tag{2}$$

On the other hand,

$$\frac{Vv}{2} = E \tag{3}$$

From (1), (2), and (3), we obtain:

$$\frac{1}{v} + \frac{1}{f} = \frac{1}{E} + \frac{1}{2} \tag{4}$$

The simple solutions of (4) are

	v	f	E	V	F
I	$v = E$	2		2	$F = E$
II	2	$f = E$		$V = E$	2

Division I has 2 vertices and as many faces as edges. Each face is bounded by two lines, $f = 2$. The number of edges is arbitrary $E = 1, 2, 3, \ldots$ (see Figure 1).

SOLUTION I

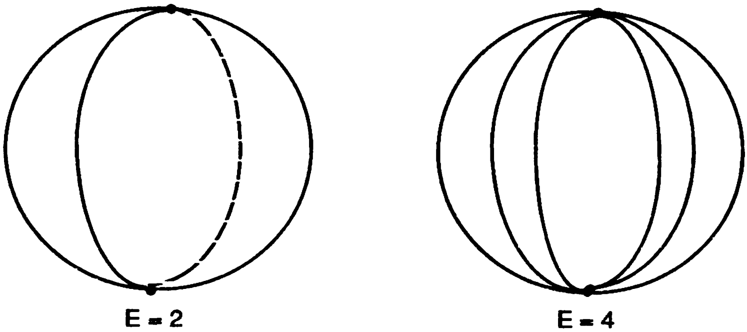


FIGURE 1

Division II has two faces and as many vertices as edges, see Figure 2. The number of edges is arbitrary $E = 1, 2, 3, \ldots$

SOLUTION II

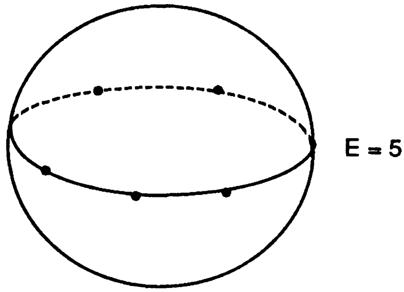


FIGURE 2

Here we have a sphere divided into two hemispheres separated by an E-gon.

● PROBLEM 9–32

Consider the equation:

$$\frac{1}{v} + \frac{1}{f} = \frac{1}{E} + \frac{1}{2} \qquad (1)$$

which describes the regular divisions of a sphere. Find all possible non-trivial solutions of (1).

SOLUTION:

For $E = 1$, we obtain trivial solutions

$$v = 1, \quad f = 2 \quad \text{or} \quad v = 2, \quad f = 1.$$

For $E = 2$, the only solution is $f = v = 2$, discussed in Problem 9-31. We are left with $E = 3, 4, 5, \ldots$ Hence,

$$\frac{1}{2} < \frac{1}{v} + \frac{1}{f} < 1 \qquad (2)$$

(2) excludes $f = 1$. Case $f = 2$ was already discussed.

The possible values of f are $f = 3, 4, 5$. Note that $f = 6, 7, 8, \ldots$ are incompatible with (2).

By the same token, the admissible values of v are $v = 3, 4, 5$.

f \ v	3	4	5
3	OK	OK	OK
4	OK		
5	OK		

The crossed-out values do not satisfy (2). Hence, we are left with five cases.

● PROBLEM 9-33

Find the number of faces F, edges E and vertices V for each of five regular polyhedra found in Problem 9-32.

SOLUTION:

In Problem 9-32, we found possible values of f and v:

v	f	E	F	V	Name
3	3	6	4	4	tetrahedron
4	3	12	8	6	octahedron
3	4	12	6	8	cube
5	3	30	20	12	icosahedron
3	5	30	12	20	dodecahedron

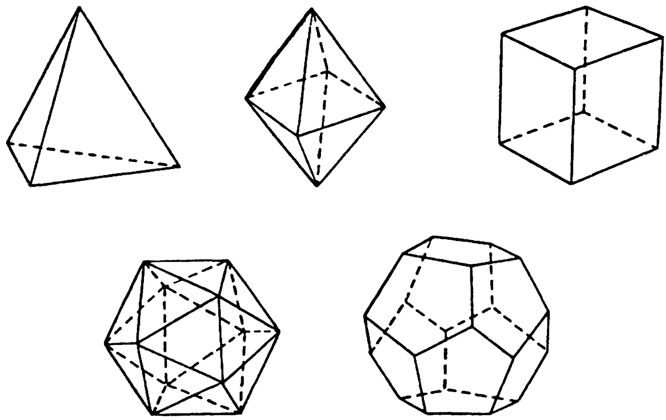
The values of E are computed from

$$\frac{1}{E} = \frac{1}{f} + \frac{1}{v} - \frac{1}{2} \tag{1}$$

The values of F and V are obtained from

$$F = \frac{2E}{f}, \qquad V = \frac{2E}{v} \tag{2}$$

Such five regular polyhedra, known as Platonic polyhedra, are depicted in the following figure.



● PROBLEM 9-34

A separation of a surface consisting of vertices linked together by edges is called a map. For any given surface, the expression

$$V - E + F = \chi \tag{1}$$

is invariant. Hence, it remains invariant for all surfaces homeomorphic to the given surface.

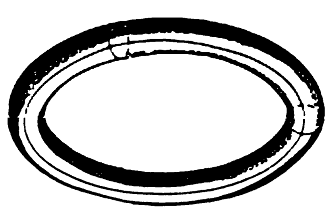


FIGURE 1

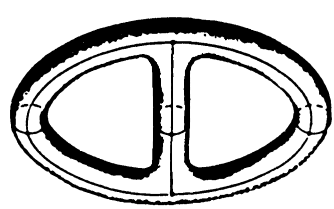


FIGURE 2

This invariant number χ is called the Euler characteristic of the surface, and it is a topological property of the surface independent of the actual values of V, E, and F. For the sphere $\chi = 2$, find χ for a torus and for a two-fold torus, see Figures 1 and 2.

SOLUTION:

The Euler characteristic remains the same for any map on a torus, so we can use the map of Figure 1.

$$V = 2 \quad E = 4 \quad F = 2. \tag{2}$$

Thus,

$$V - E + F = 2 - 4 + 2 = 0. \tag{3}$$

For torus, $\chi = 0$.

From Figure 2, for a two-fold torus

$$V = 5 \quad E = 9 \quad F = 2. \tag{4}$$

Hence, for a two-fold torus

$$\chi = 5 - 9 + 2 = -2. \tag{5}$$

● PROBLEM 9–35

1. Figure 1 depicts regular polyhedra. Find the Euler characteristic for each one.

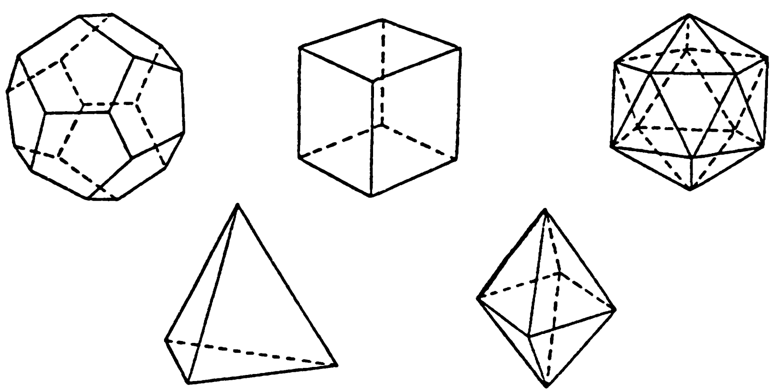


FIGURE 1

2. Is it possible

a) to draw a map on a sphere having seven vertices linked by ten arcs and defining four regions?

b) to draw a map on a two-fold torus having five vertices linked by eight arcs and defining two regions?

SOLUTION:

1. Consider the tetrahedron

$$V = 4 \quad E = 6 \quad F = 4.$$

Hence,

$$V - E + F = 4 - 6 + 4 = 2. \tag{1}$$

We can perform calculations for all polyhedra and in each case, obtain $\chi = 2$. This result is obvious and does not require any computing. Any simple polyhedron can be continuously deformed into a sphere; hence, $\chi = 2$.

2. a) For a sphere

$$V - E + F = 2$$

$$7 - 10 + 4 = 1 \neq 2 \tag{2}$$

Not possible.

b) For a two-fold torus

$$V - E + F = -2$$

$$5 - 8 + 2 = -1 \neq -2 \tag{3}$$

It is not possible to draw such a map.

● PROBLEM 9-36

Use the triangulation method to prove Euler's formula

$$V - E + F = 2 \tag{1}$$

for a cube.

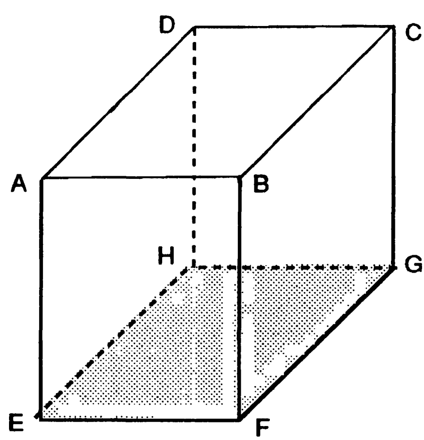


FIGURE 1

SOLUTION:

From the cube $ABCDEFGH$, we have to remove any face, say $EFGH$. The Euler characteristic $\chi = V - E + F$ for a cube is now decreased by 1 since V

and E remain constant and F decreases by 1. The cube with the removed face is transformed until all vertices lie in a plane (see Figure 2).

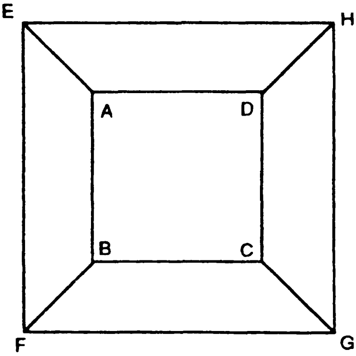


FIGURE 2

This deformation does not change χ. Now we shall carry out the triangulation as shown in Figure 3.

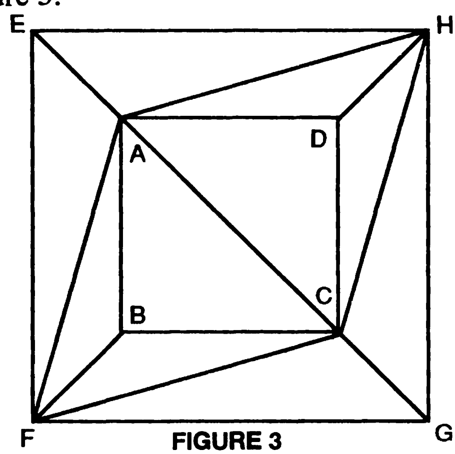


FIGURE 3

We have $V = 8, E = 17, F = 10$.

Note that whenever we add a triangle, V remains constant, whereby E and F increase by 1. Hence, χ remains the same.

Triangles are now removed until one remains; e.g., ABC, for which

$$V = 3, \quad E = 3, \quad F = 1$$

Hence,

$$V - E + F = 1$$

for a triangle and

$$V - E + F = 1 + 1 = 2$$

for a cube.

The value of $V - E + F$ remains invariant within any topological equivalence class.

Describe the method of triangulation applied to prove Euler's formula

$$V - E + F = 2$$

for polyhedra.

SOLUTION:

From a polyhedron with V vertices, E edges, and F faces we remove one face. Removal of one face decreases $V - E + F$ by one.

The new surface is deformed until all vertices and edges lie in a plane. We obtain something that is topologically equivalent to a disc. Now we perform the triangulation. Each face is divided into a triangle in such a way that no new vertices are created (Figure 1).

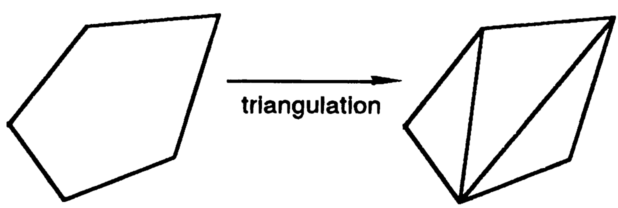


FIGURE 1

Note that $V - E + F$ remains unchanged.

We shall remove the triangles in such a way that the triangle removed has at least one edge on the boundary. Two situations are possible (see Figure 2).

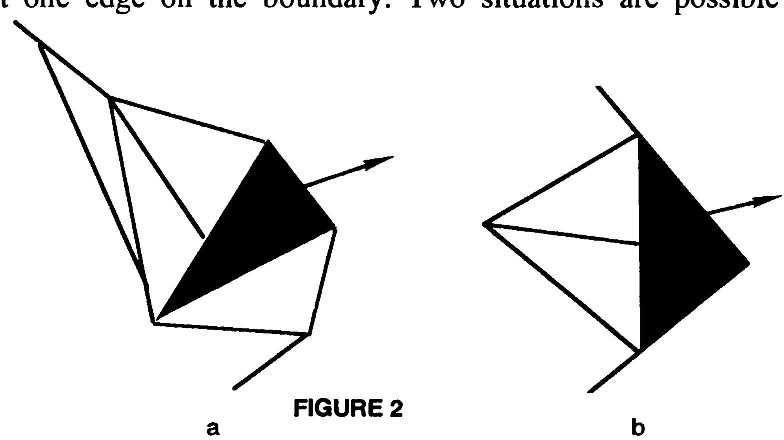


FIGURE 2

For a) V remains the same, E and F are decreased by one; hence, $V - E + F$ is invariant.

Similarly, for b) $V - E + F$ is invariant because V and F decrease by one and E decreases by two. Finally we are left with one triangle for which

$$V - E + F = 3 - 3 + 1 = 1.$$

Therefore, for a polyhedron, $\chi = 2$.

● PROBLEM 9–38

Verify the expression

$$V - E + F = \chi \tag{1}$$

for the drawings depicted in Figure 1 and Figure 2.

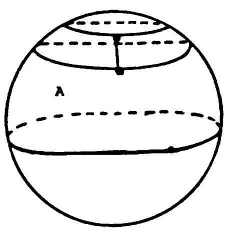


FIGURE 1

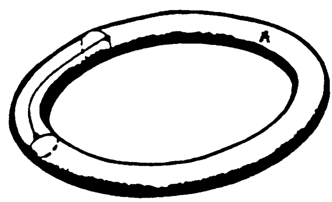


FIGURE 2

Explain any discrepancies.

SOLUTION:

For a sphere

$$V - E + F = 2. \tag{2}$$

From Figure 1

$$V = 3, \quad E = 4, \quad F = 4. \tag{3}$$

Hence,

$$V - E + F = 3 - 4 + 4 = 3 \neq 2. \tag{4}$$

All regions of a map should be simply connected. Region A is not simply connected. Thus rule (1) cannot be applied.

For a torus $\chi = 0$. From Figure 2

$$V = 2, \quad E = 3, \quad F = 2. \tag{5}$$

and

$$V - E + F = 1 \neq 0 \tag{6}$$

because A is not simply connected.

● PROBLEM 9-39

Each surface is characterized by two numbers

1. the genus, g

2. the Euler characteristic, χ.

What is the relationship, if any, between these numbers?

SOLUTION:

The genus of the surface is the greatest number of distinct continuous non-self-intersecting closed curves which may be drawn on a surface without separating it into distinct regions. Each division of a surface to become a map, has to include some edges so that all regions are simply connected. That is related to the genus of the surface.

	Genus	Euler characteristic
Sphere	0	2
Torus	1	0
Two-fold Torus	2	-2
n-fold Torus	n	$2-2n$

The relation between the genus and the Euler characteristic of any given surface is

$$V - E + F = \chi = 2 - 2g.$$

● PROBLEM 9-40

Suppose you have a "hairy" sphere and a "hairy" torus. What happens when you comb down the hair on a sphere? Is it possible to comb a torus nicely?

SOLUTION:

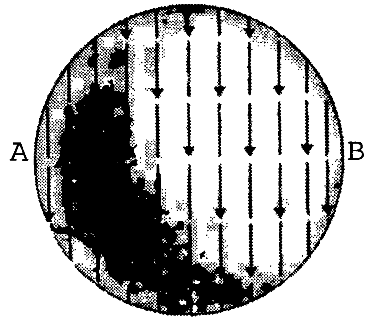


FIGURE 1

Figure 1 depicts a sphere with its hair combed. This "hairdo" has two points of discontinuity, A and B, as illustrated in Figure 2.

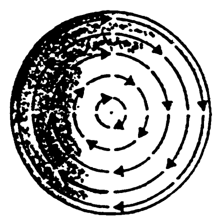


FIGURE 2

Another way of brushing a sphere is shown in Figure 3, which has two points of discontinuity C and D.

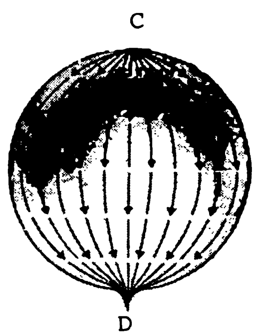


FIGURE 3

It is possible to brush the hair of a torus in a nice way; that is, in such a way that continuity is preserved everywhere (see Figure 4).

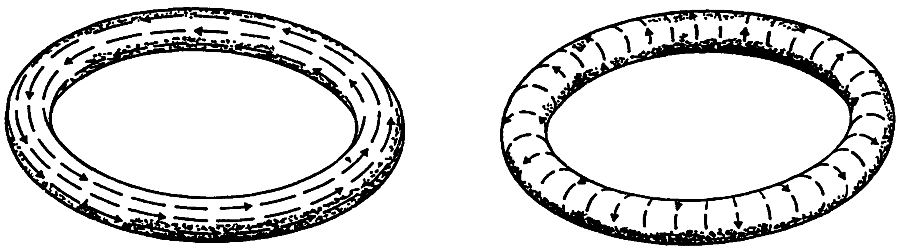


FIGURE 4

Note that all this "brushing" is equivalent to assigning a direction to each point of a surface.

This problem belongs to differential topology.

● PROBLEM 9-41

> Instead of combing hair, we can visualize a flow of fluid on the surface. The points of discontinuity are called singular points. Make a sketch of a few singular points.

SOLUTION:

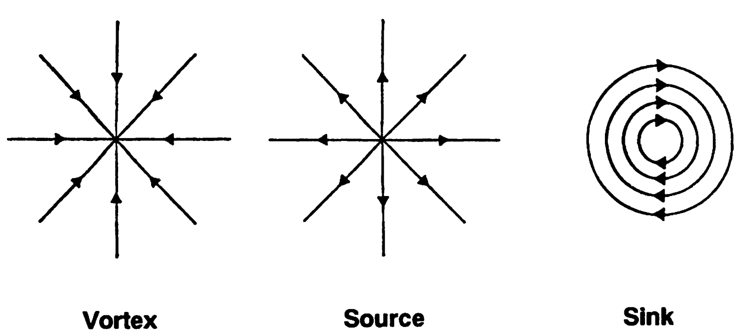
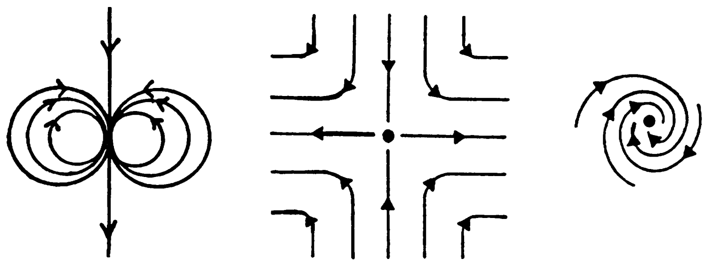


FIGURE 1

Figure 1 depicts some often encountered singular points. Some other singular points are illustrated in Figure 2.

In most cases, the name of a singular point explains its nature.



Dipole Crosspoint Focus

FIGURE 2

● PROBLEM 9-42

For each singular point described in Problem 9-41, find its index.

SOLUTION:

To each singular point, we assign an integer called its index, which is obtained by travelling around this point along a circle in a counter-clockwise direction and counting the number of counter-clockwise revolutions made by an arrow with its base on the path and pointing in the direction of the flow on the surface.

Let us compute the index of a source.

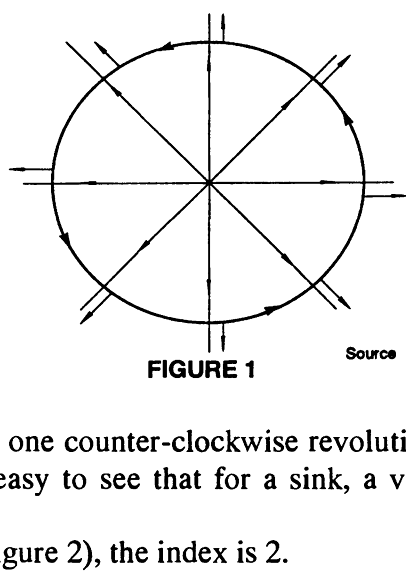


FIGURE 1 Source

The little arrow makes one counter-clockwise revolution, hence the index of a source is one. It is easy to see that for a sink, a vortex and a focus, the index is one.

For a dipole (see Figure 2), the index is 2.

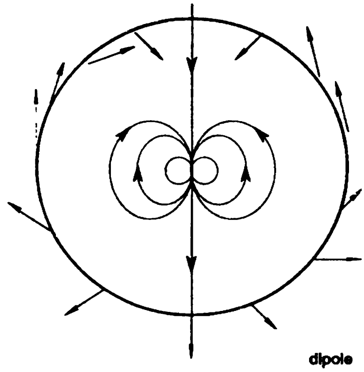


FIGURE 2

For the crosspoint (see Problem 9-41) the index is – 1. It is easy to see that the little arrow makes one clockwise revolution.

● PROBLEM 9-43

Brush a hairy sphere so that it has only one singular point. Make a sketch and determine the index of this singularity.

SOLUTION:

It is possible to brush a sphere in such a way, that it would have only one singularity, namely a dipole.

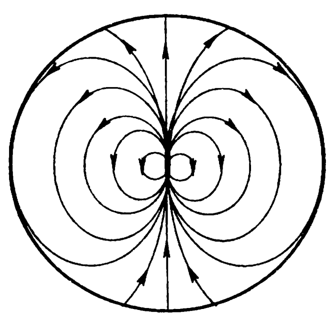


FIGURE 1

The index of a dipole is two.

● PROBLEM 9-44

THEOREM

The Euler characteristic of the surface is equal to the sum of the indices

of the singular points of this surface.

■

Verify this theorem for the surfaces shown in Problems 9-40 and 9-43.

SOLUTION:

Consider the surface of a sphere depicted in Figure 1 of Problem 9-40. It has two vortices. The index of a vortex is one. Hence,

$$1 + 1 = 2$$

which is the Euler characteristic of a sphere.

Similarly, Figure 3 of a sphere of Problem 9-40 has two singular points: a sink with index one and a source with index one. Again,

$$1 + 1 = 2.$$

Both toruses of Figure 4 of Problem 9-40 have no singular points. Hence, the sum of indices is 0 which is the Euler characteristic of a torus.

The sphere depicted in Problem 9-43 has one dipole, whose index is 2.

● PROBLEM 9-45

There is some fluid flow on the sphere which contains a crosspoint. Complete the description of this flow. What other singularities contain a flow with a vortex on a torus?

SOLUTION:

The sum of the indices of the singularities on a sphere is two. The index of a crosspoint is –1. Thus,

$$-1 + \Sigma \text{ indices of other singular points} = 2.$$

Here are some possible combinations:

sink and two sources

source and two sinks

dipole and a sink.

The Euler characteristic of a torus is 0. So, if there is a vortex with index 1, there must be a crosspoint with index – 1.

We obtain

$$1 + (-1) = 0.$$

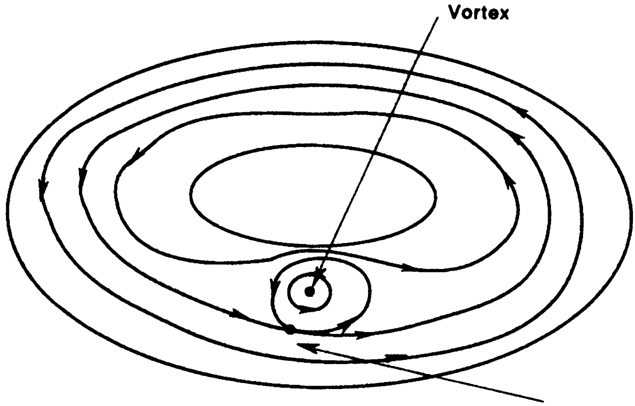


FIGURE 1

● PROBLEM 9-46

Show that the sum of the indices of singular points of a surface is equal to its Euler characteristic.

SOLUTION:

For any map on a given surface the expression

$$V - E + F$$

is invariant and equal to its Euler characteristic. Thus

$$V - E + F = \chi.$$

Suppose there is a map drawn on some surface. This map can be replaced by a flow as follows:

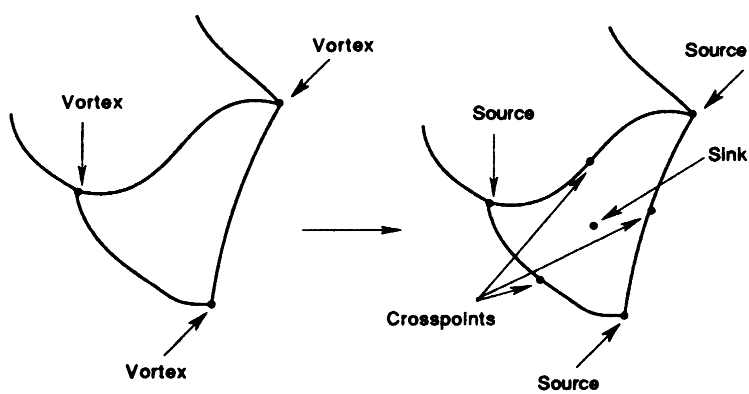


FIGURE 1

1. Replace vertices by sources.

2. Put crosspoints at the center of each arc.

3. Put a sink at the center of each region.

We obtain:

$$V \text{ sources of index } 1$$

$$E \text{ crosspoints of index } -1$$

$$F \text{ sinks of index } 1$$

The sum of indices is

$$V - E + F$$

■

● PROBLEM 9-47

Can you draw these figures without lifting the pencil and passing once and only once through each line. If not, explain why.

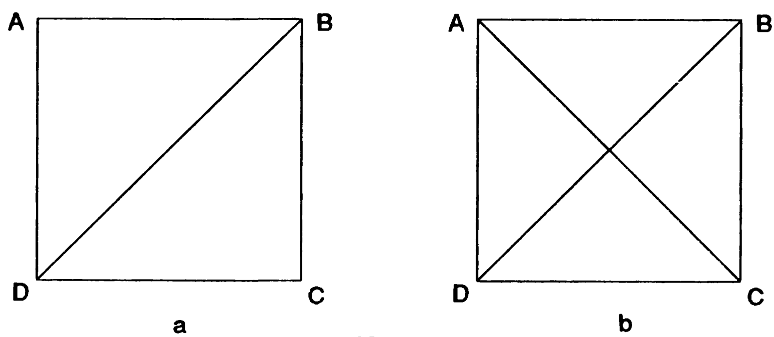


FIGURE 1

SOLUTION:

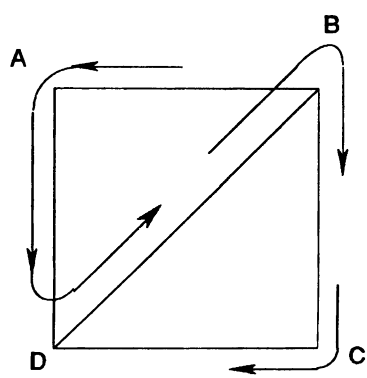


FIGURE 2

Figure 1a can be easily drawn according to the requirements as shown in Figure 2.

We can start from *B* and then move to *A, D, B, C, D*.

It is not possible to draw Figure 2b. Let us return to Figure 2. There are two points *B* and *D* where three lines merge. Then, to make a drawing we have to start with one such point and end with the other. In case of Figure 1b, we have four points, *A, B, C, D* where three lines merge. One can be used to start a drawing, the second to end it; but there is no way we can "get rid" of the remaining two points.

● PROBLEM 9–48

Determine the order of each vertex.

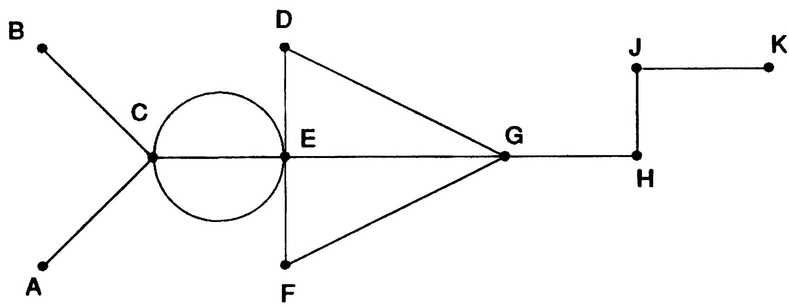


FIGURE 1

Which vertices are odd and which are even?
Why must the total number of arc-ends in any network be even?
Show that the total number of odd vertices must be even.

SOLUTION:

A network consists of a finite number of vertices linked by arcs. The arcs must be non-intersecting. Such a network is sometimes called a linear graph. The number of arc-ends meeting at the vertex is called an order.

A vertex whose order is an even number is called an even vertex. A vertex with an odd order is called an odd vertex.

From the drawing we have

A	1	odd
B	1	odd
C	5	odd
D	2	even

E	6	even
F	2	odd
G	4	even
H	2	even
J	2	even
K	1	odd

The number of arc-ends is twice the number of arcs; hence, it is even.

The number of arc-ends is equal to the sum of the orders of all the vertices. Hence, the total number of odd vertices must be even.

● PROBLEM 9–49

Show that the network

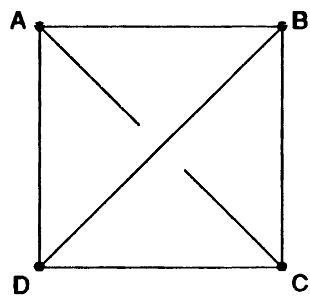


FIGURE 1

is planar.

SOLUTION:

A network that can be mapped onto a simply connected surface in such a way that the arcs don't intersect is called planar.

The network of Figure 1 can be mapped on the plane (see Figure 2).

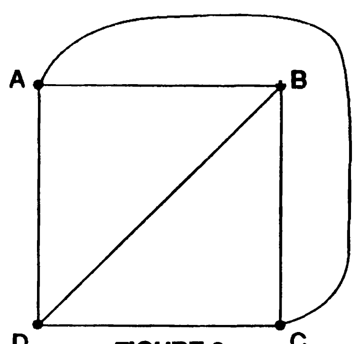


FIGURE 2

This network can also be mapped onto a sphere as shown in Figure 3.

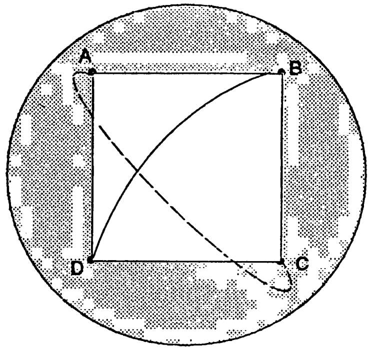


FIGURE 3

● PROBLEM 9–50

Suppose there are three houses, A, B, and C. Each house has to be connected with water, electricity and gas in such a way that the pipes do not intersect. Can it be done?

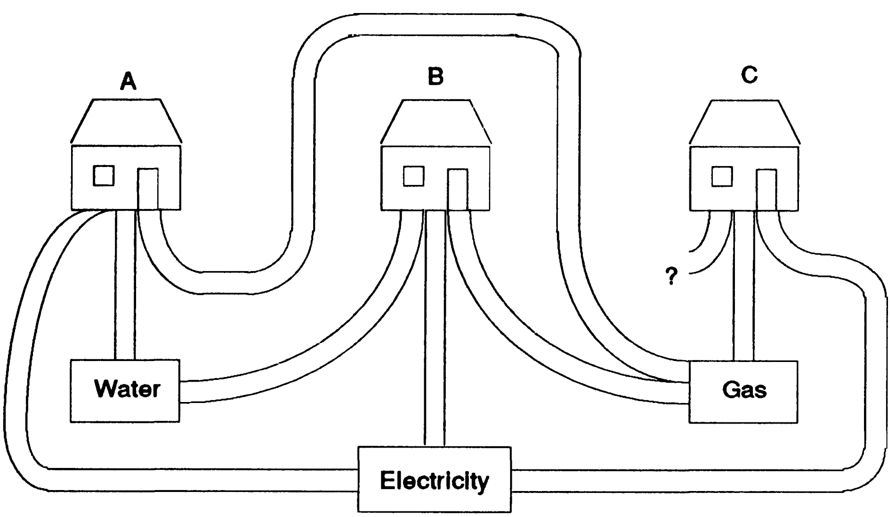


FIGURE 1

SOLUTION:

It cannot be done. Here is why.

Let us connect the first house A with water, electricity, and gas, and then proceed through these points to the second house B, see Figure 2.

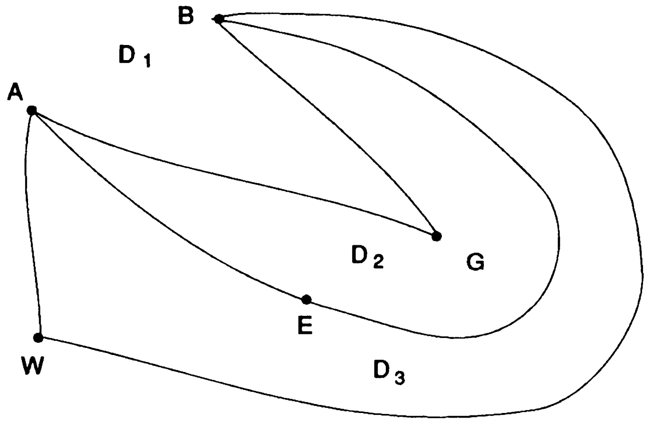


FIGURE 2

We have three lines from A to B which do not cross. These lines divide the whole plane into three areas: D_1, D_2, D_3. The third house C lies in one of the areas.

If it lies in D_1, then E cannot be connected; if in D_2, then W cannot be connected; and if in D_3, then G cannot be connected.

● PROBLEM 9-51

Give an example of a non-planar network.

SOLUTION:

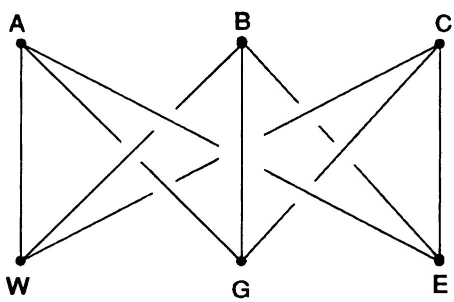


FIGURE 1

Note that the network depicted in Figure 1 illustrates Problem 9-50.

Another example is the complete network on five vertices shown in Figure 2.

It can be proven that every non-planar network must contain either a network of Figure 1 or network of Figure 2 as a sub-network.

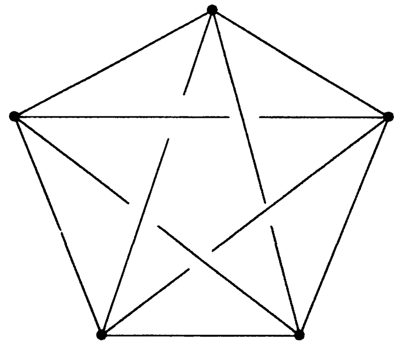


FIGURE 2

A network is complete when all vertices are directly linked to each other by the minimum number of arcs.

● PROBLEM 9–52

Give an example of a network that cannot be traversed by a single path.

SOLUTION:

A sequence of arcs which can be followed continuously without any arc being passed more than once is called a path. Figure 1 depicts a path, *ABCD*.

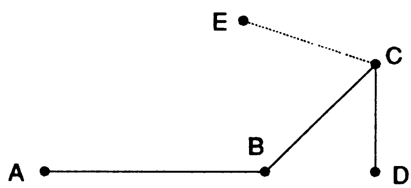


FIGURE 1

A path traverses a network if every arc of the network is included in the path.

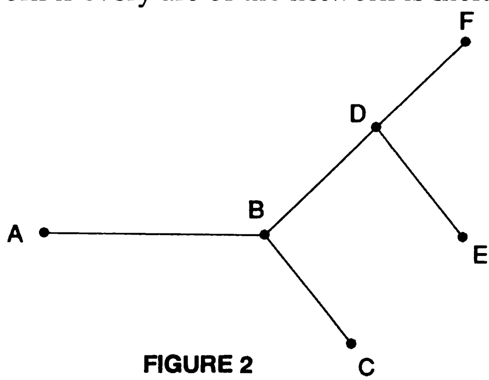


FIGURE 2

No path traverses a network shown in Figure 2.

A closed path (that is such, which starts and finishes at the same vertex)

is called a circuit.

A path that is not closed is called open.

● PROBLEM 9-53

Show that a network which has more than two odd vertices cannot be traversed by a single path.

SOLUTION:

Figure 1 depicts a network, which

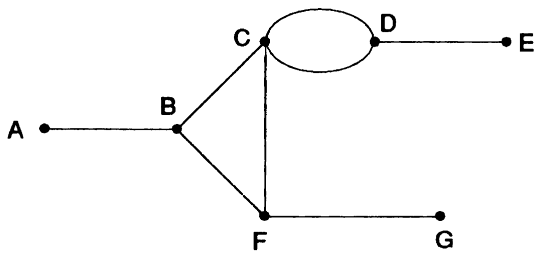


FIGURE 1

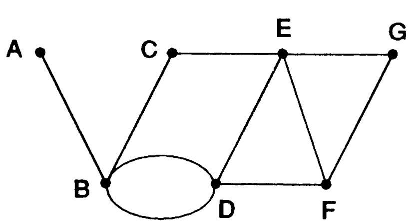
cannot be traversed by a single path.

This network has three vertices of the order of one, three vertices of the order of three and one vertex of the order of four. If a path traversing a network is closed, then an even vertex may be the starting and terminating point of the "journey." If a path traversing a network is open, then it starts and terminates at odd vertices. Hence, we conclude that if a network has more than two odd vertices, it cannot be traversed by a single path.

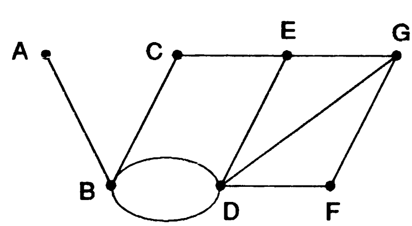
The network in Figure 1 has six odd vertices and cannot be traversed by a single path.

● PROBLEM 9-54

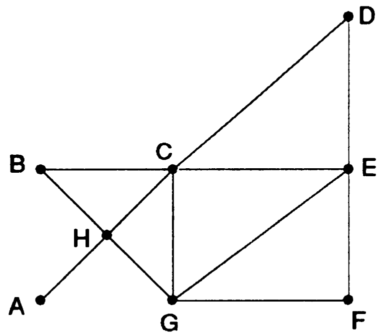
Which of the following networks can be traversed by a single path?



(a)



(b)



(c)

FIGURE 1

SOLUTION:

A connected network can be traversed by a single open path, if and only if it has exactly two odd vertices.

Network (a) of Figure 1 has two odd vertices A and F. It can be traversed *ABDBCEDFGEF*. Network (b) of Figure 1 has four odd vertices A, E, G, D. It cannot be traversed by a single path.

Network (c) of Figure 1 has two odd vertices A and C, and it can be traversed by a single path *AHCDEFGHBCGEC*.

● PROBLEM 9–55

Which of the networks is connected and which is disconnected?

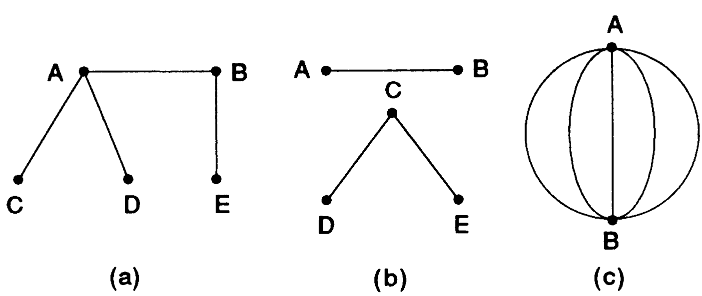


FIGURE 1

Find a necessary condition for a network to be connected.

SOLUTION:

A network is said to be connected if every pair of vertices belongs to some path; otherwise the network is disconnected.

Networks (a) and (c) are connected.

Network (b) is disconnected, vertices A and C don't belong to any path. Let a be the number of arcs and n the number of vertices. Then a necessary condition for a network to be connected is

$$n - 1 \leq a.$$

This condition is necessary but not sufficient.

● PROBLEM 9–56

Reduce a network depicted in Figure 1 to a tree.

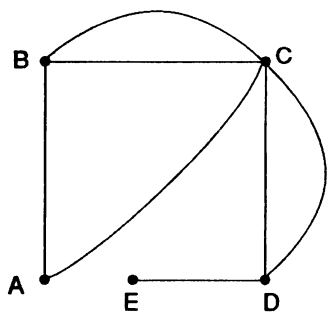


FIGURE 1

SOLUTION:

Any connected network, such that the number of its vertices is one more than the number of its arcs is called a tree. Any connected network can be reduced to a tree by the removal of appropriate arcs. We can remove BC, AC, and CD to obtain a network (see Figure 2) which is a tree.

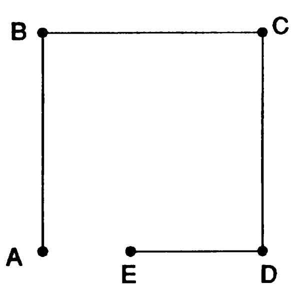


FIGURE 2

● PROBLEM 9–57

A planar network is mapped onto a simply connected surface, so that no arcs intersect.

The number of arcs and vertices is given. Determine the number of bounded regions into which the surface is separated.

SOLUTION:

Let v be the number of vertices and a the number of arcs. Then the number of bounded regions is

$$a - n + 1.$$

Indeed, for a triangle

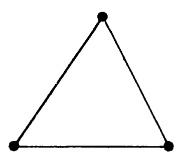


FIGURE 1

we have $3 - 3 + 1 = 1$. For an arc with two vertices $1 - 2 + 1 = 0$.

FIGURE 2

According to the network depicted in Figure 2, by adding new arcs and vertices, we can obtain any network.

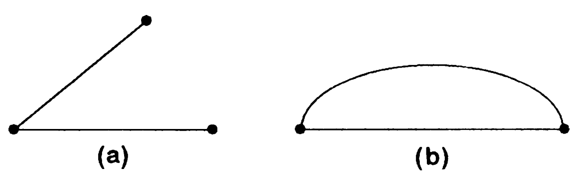


FIGURE 3

In Figure 3(a) we added one arc and one vertex, hence the number $a - n + 1$ remains the same. In Figure 3(b) one arc was added and the number of vertices is constant, hence $a - n + 1$ increases by 1.

In such a way, we can obtain any network. That completes the proof.

● PROBLEM 9–58

List the tie-sets, which specify the network in Figure 1.

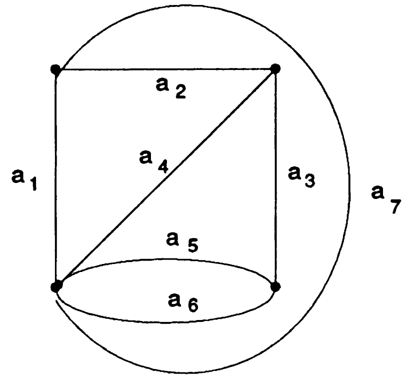


FIGURE 1

SOLUTION:

Any planar network can be completely determined in terms of its independent bounded regions or in terms of paths between its independent pairs of vertices. There are $\frac{V(V-1)}{2}$ pairs but only $v-1$ independent pairs (v is the number of vertices). A network is defined by tie-sets when its structure is determined by paths between its independent pairs of vertices. A tie-set is a single closed path, such that exactly two arcs meet at each vertex. Figure 2 depicts some tie-sets.

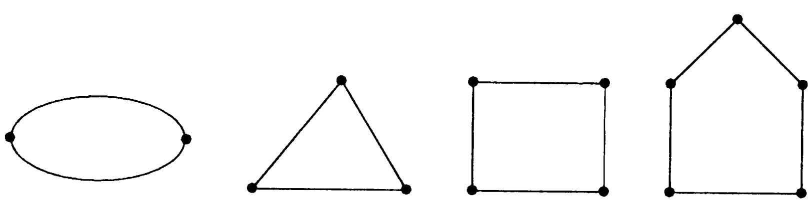


FIGURE 2

Here is one of of possible lists of tie-sets of the network of Figure 1

$$a_5\, a_6,\ a_1\, a_2\, a_4,\ a_3\, a_5\, a_4,\ a_2\, a_3\, a_6\, a_7.$$

Another possibility is

$$a_1\, a_2\, a_3\, a_6,\ a_1\, a_4\, a_3\, a_6\, a_7,\ a_3\, a_4\, a_6,\ a_1\, a_2\, a_3\, a_5$$

or any other combination of four independent closed paths.

• PROBLEM 9–59

Is it possible to traverse a network depicted in Figure 1 by a single path?

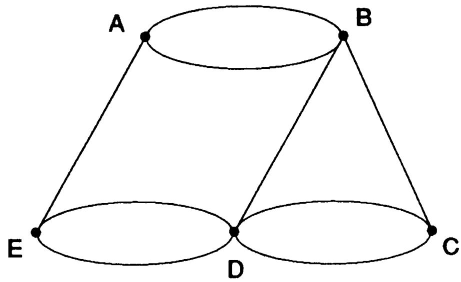


FIGURE 1

SOLUTION:

To be traversed by a single path, a network has to be connected (it is a necessary condition but not sufficient).

If a network has two odd vertices, then these vertices have to be the initial and final vertices of the path. Now suppose a network has an even number $2n > 2$ of odd vertices. Then at least n paths are necessary to traverse it.

The network of Figure 1 has four odd vertices, hence two paths are needed to traverse this network. We can choose for example

$$ABAEDBCDC \quad \text{and} \quad ED.$$

● PROBLEM 9-60

Here is the famous problem of the Königsberg bridges solved by Euler. Is it possible to cross them all in turn, passing no more than once over each?

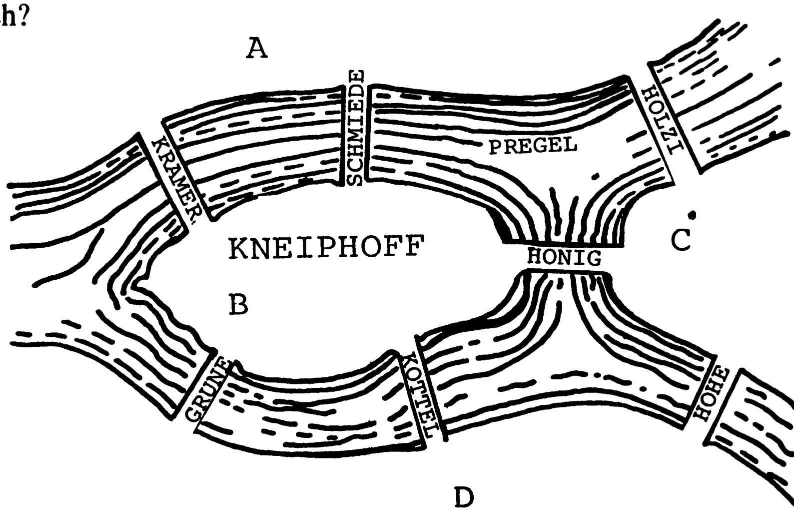


FIGURE 1

SOLUTION:

Let us denote the regions as shown in Figure 1.

Then we have to traverse a network shown in Figure 2 with one single path.

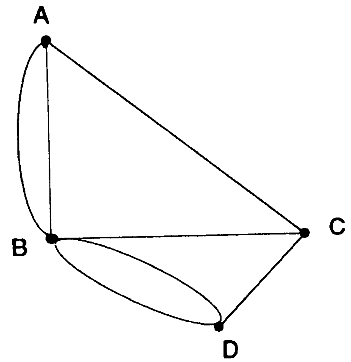


FIGURE 2

But this is impossible, because A, B, C, and D are odd vertices.

● PROBLEM 9–61

> Consider again the Königsberg bridges. Where would the eighth bridge be built in order to make the walk possible? Again, we have to cross all bridges by passing only once over each.

SOLUTION:

From Figure 2 of Problem 9-60, we see that there are four odd vertices.

Hence the eighth bridge has to eliminate two odd vertices, for example, C and D (see Figure 1)

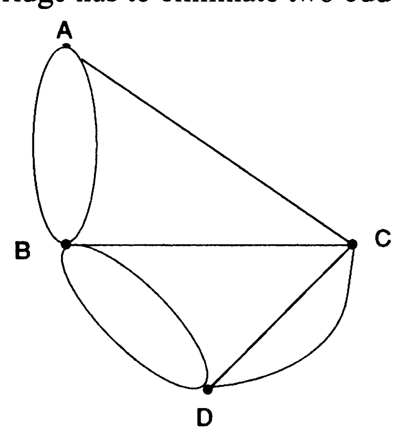


FIGURE 1

We have to build a bridge between C and D as shown in Figure 2.

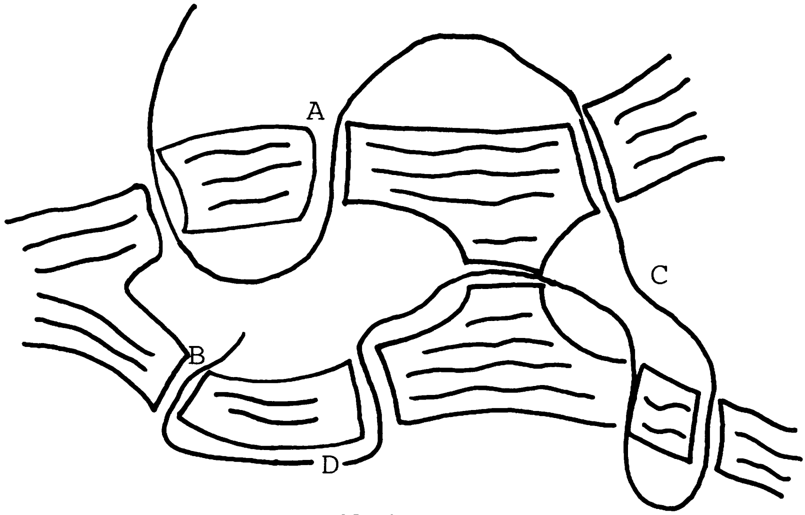


FIGURE 2

The only two odd vertices are now A and B. Hence a walk has to start at A (or B) and end at B (or A).

● PROBLEM 9-62

Is it possible to draw the figure shown in Figure 1 without lifting the pencil and passing once and only once through each point?

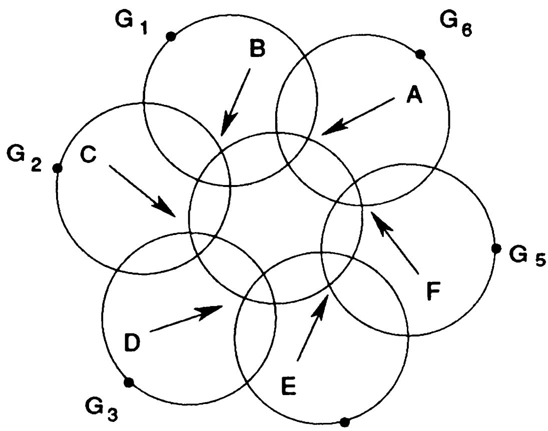


FIGURE 1

SOLUTION:

Yes. Note that all the vertices are of the order of four.

We can start from A and draw a circle $A\ G_1\ A$, then move to B and draw a circle $B\ G_2\ B$, then move to C and so on, until we draw the last surrounding circle $F\ G_6\ F$. To complete the drawing we sketch FA, which closes the inner circle.

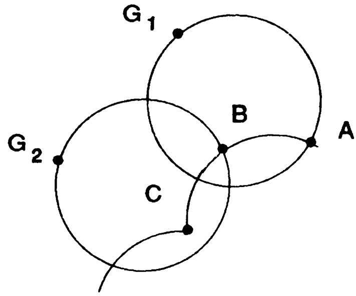


FIGURE 2

It is also possible to draw the Olympic symbol (see Figure 3) without lifting the pencil and passing only once through each point.

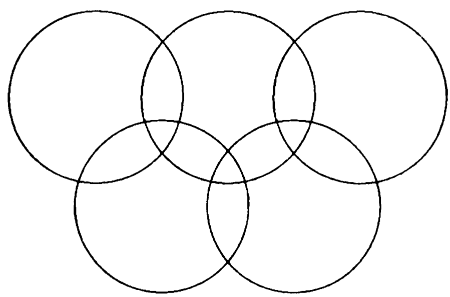


FIGURE 3

● PROBLEM 9–63

Here is an odd puzzle. It's quite possible that you already know the solution. If you don't, try to find it.

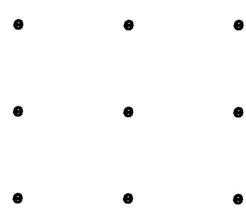


FIGURE 1

The network consists of 9 dots (see Figure 1). Draw 4 lines without lifting the pencil from the paper so as to cross out every dot.

SOLUTION:

To solve the problem, we have to leave the frame of the dots.

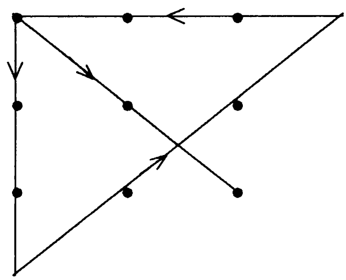


FIGURE 2

Another solution is shown in Figure 3. Here the line crosses each dot once and only once.

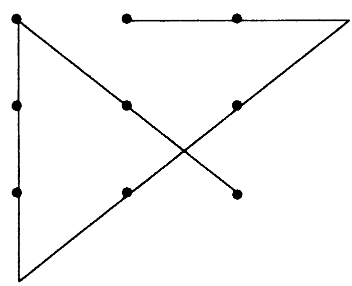


FIGURE 3

● PROBLEM 9-64

Suppose you own a mansion with the floor-plan depicted in Figure 1.

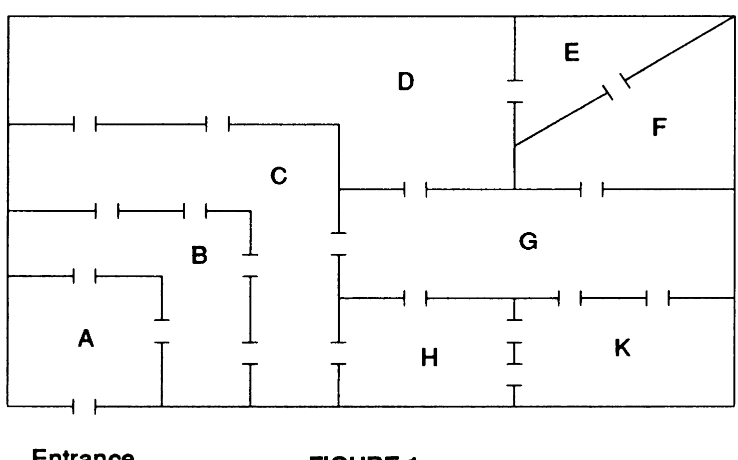


FIGURE 1

To support yourself, you have to guide tourists through each room. Is is possible to traverse the mansion starting at the entrance and passing each door once and only once?

SOLUTION:

Let us denote the rooms *A, B, C, D, E, F, G, H,* and *K*.

Each room represents a vertex of a network. The number of doors is the order of the vertex, hence A has order 3, $B - 4$, $C - 4$, $D - 4$, $E - 2$, $F - 2$, $G - 6$, $H - 4$, $k - 4$. There is only one odd vertex. It is possible to traverse the network by a single path. One possible route is shown in Figure 2.

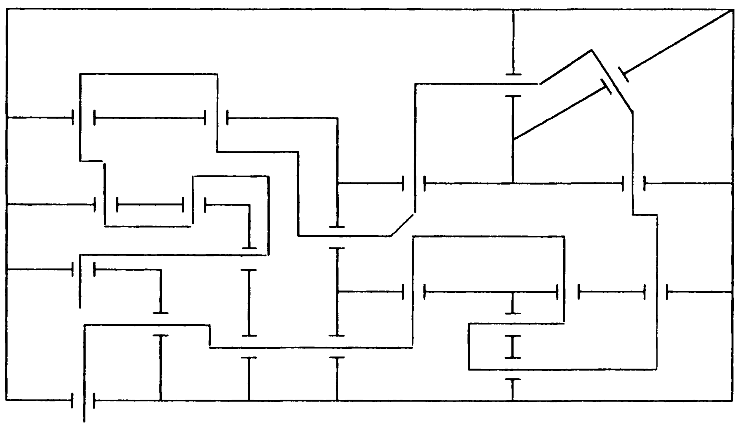


FIGURE 2

● PROBLEM 9–65

Prove the following theorem:

THEOREM

Every network with all even vertices, or with, at the most, two odd vertices can be traversed by a single path.

SOLUTION:

Consider first, the case when all vertices are even.

We can start at an arbitrary vertex A. Since A is even, the path must end at A. If parts of the network not already visited remain and the order of A is higher than 2, we can draw another closed path.

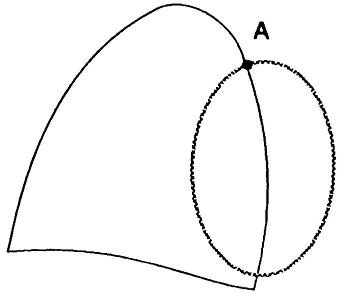


FIGURE 1

Two closed paths with a common point A can be considered as one closed path.

If the task is not yet accomplished, the new path must be linked at B with the rest of the network.

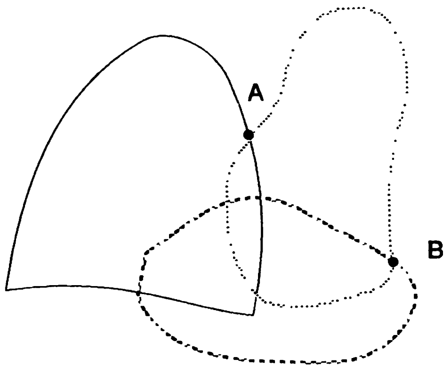


FIGURE 2

We can repeat this procedure until we reach the solution, that is when the path traverses the whole network.

The reasoning is almost the same for the case of a network with two odd vertices A and B.

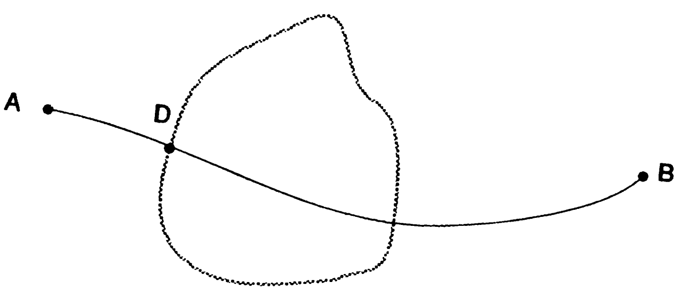


FIGURE 3

We start at A (or B) and end at B (or A). The rest of the network contains only even vertices previously discussed and is linked to AB at some point D.

What regular polygons with all diagonals can you draw with one stroke, that is, without lifting a pen and passing once and only once through each one?

SOLUTION:

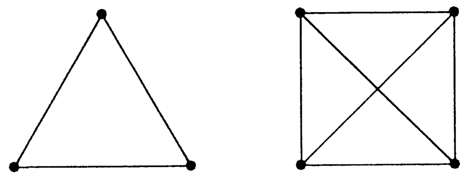
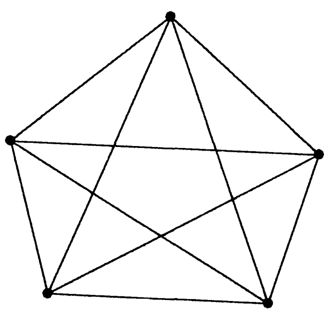


FIGURE 1

It is possible to draw a triangle but not a square, which has four odd vertices.

For a corresponding network to have only even vertices, the polygon must have an odd number of sides. Figure 2 depicts 5-gon and 7-gon.



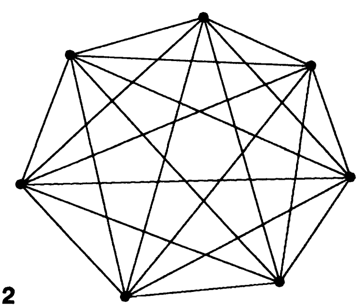


FIGURE 2

If you have enough patience you can draw polygons of higher order 19-, 21-, 23-, 25-gons etc. Figure 3 illustrates 23-gon with all of its diagonals.

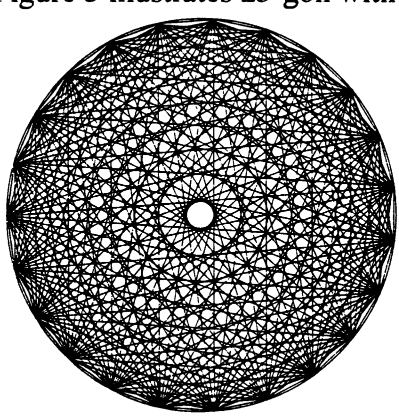


FIGURE 3

● PROBLEM 9-67

Suppose you have three chains with clips. Arrange them in such a way that they interlock as a whole but individual pairs do not interlock. That is, if you open any one chain, they all become separated.

SOLUTION:

Figure 1 does not offer a solution.

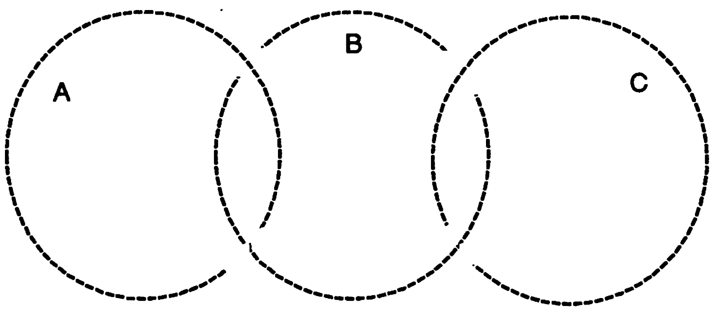


FIGURE 1

Opening *B* separates all of the chains. But opening *A or* C does not. Figure 2 illustrates the solution.

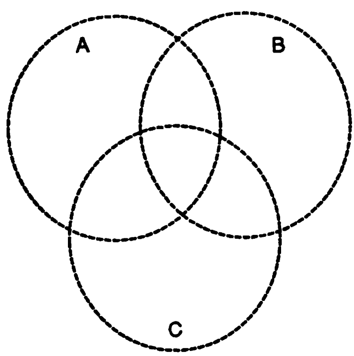


FIGURE 2

Opening any chain separates the remaining two.

The knowledge of knots is part of topology. It has been proven that it is impossible to tie two knots at two ends of a rope in such a way that when brought together they may cancel each other.

● PROBLEM 9-68

Prove that five (or more) countries are never neighbors of each other.

SOLUTION:

It is easy to show that four countries can be neighbors of each other, see Figure 1.

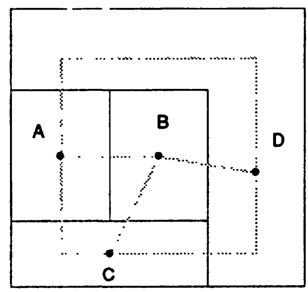


FIGURE 1

Let A, B, C, and D be the capitals or respective countries. Their capitals can be connected by roads that pass through only two countries (Figure 1). Connecting the capitals A, B, and C we get the triangle shown in Figure 2.

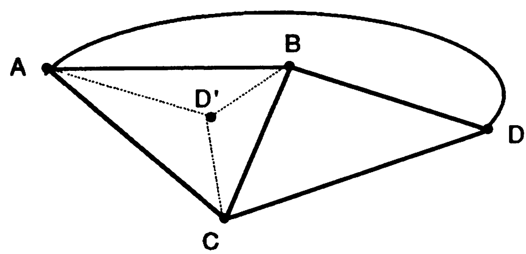


FIGURE 2

D can be interior or exterior of this triangle. In both cases, we obtain a large triangle composed of three small adjacent triangles. E is the capital of the fifth country. It lies in one of the small triangles or outside the large triangle, Figure 3.

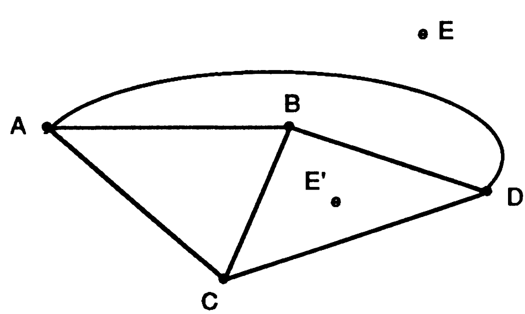


FIGURE 3

In either case, E is separated from one capital by a triangle of roads. To get from E to B (in our case) one has to cross at least one existing road.

Hence, countries with capitals B and E are not neighbors. That completes the proof.

● PROBLEM 9-69

Show that the chromatic number of a plane surface is larger than three.

SOLUTION:

Here we deal with the famous four-color problem. In 1976 it was proven that:

every map drawn on a sheet of paper can be colored with only four colors in such a way that countries having a common border receive different colors.

This problem was formulated by Francis Guthrie in 1852.

The least number of colors required to color a map on any given surface is called the chromatic number of that surface. A chromatic number is a topological property of the surface.

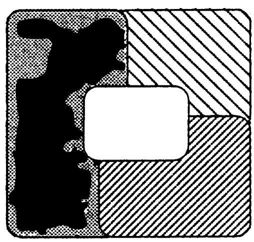


FIGURE 1

The drawing shows that three colors are insufficient to color a map on the plane surface. We will prove later that five colors are sufficient to color any map. So far we know that:

chromatic number of a plane surface ≥ 4.

● PROBLEM 9–70

Derive the regular map with the least number of additional vertices and arcs from the non-regular map depicted in Figure 1.

SOLUTION:

A map is a network consisting of a number of vertices and non-intersecting arcs linked together in a such a way that an area is separated into simply connected regions. Figures 1 depicts a map.

A map is called regular, if

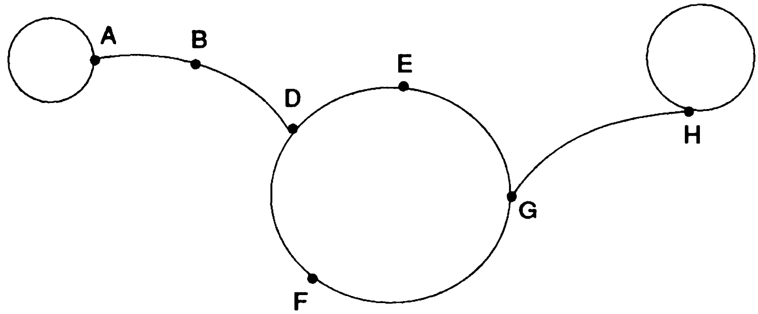


FIGURE 1

1. its vertices are of order three or more.

2. each arc separates two distinct regions.

3. each arc joins two distinct vertices.

The map of Figure 1 is not regular. We shall add some vertices and arcs to make it regular.

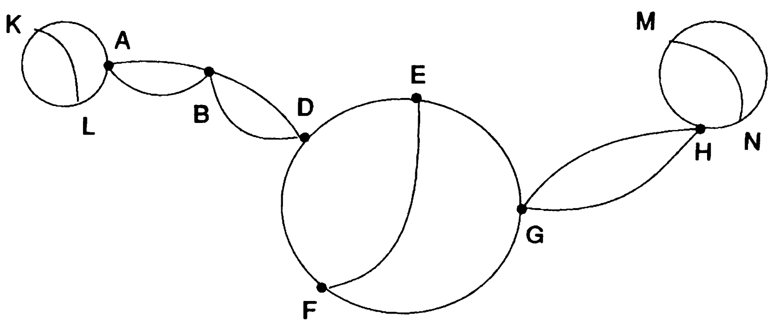


FIGURE 2

All vertices of the map of Figure 2 are of an order higher than or equal to three. The new map is regular.

● PROBLEM 9-71

> In problem 9-68, we proved that it is not possible for five countries to be located in such a way that each of them is adjacent to each of the other four. This result led Augustus DeMorgan (who was the first one to prove it) to believe that he solved the four-color problem. Why was he wrong?

SOLUTION:

The fact that five mutually adjacent countries do not exist on a map does not constitute a proof of the four-color conjecture. Quite a few mathematicians have made this mistake. Consider a map of six countries.

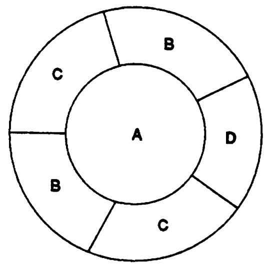


FIGURE 1

Among the six countries, there is no collection of four in which each member is adjacent to the other three. Nevertheless it takes four colors to color this map. From the figure we see that the number of colors required for a map (here 4) is not the same as the maximum number of mutually adjacent countries (here 3).

● PROBLEM 9–72

Prove the following:

THEOREM

Any regular map on the plane surface or the surface of a sphere must have at least one region bounded by fewer than six arcs.

SOLUTION:

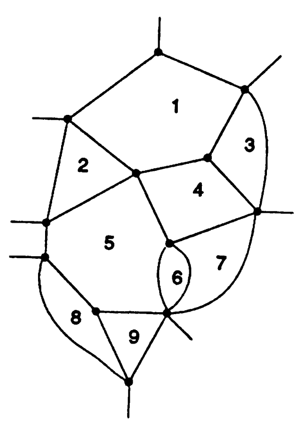


FIGURE 1

Let χ be the Euler characteristic, then

$$V - E + F = \chi \tag{1}$$

where V is the number of vertices, E, arcs, and F regions. Hence, for any surface of the Euler characteristic $\chi > 0$, we have

$$V - E + F > 0 \quad \text{or} \quad 6V - 6E + 6F > 0. \tag{2}$$

Since the map is regular, (see Problem 9-70) all vertices are of the order of 3 or higher.

$$2E \geq 3V \tag{3}$$

From (2) and (3), we obtain

$$6F - 2E > 0 \tag{4}$$

Let F_n denote the number of regions, each bounded by exactly n arcs. In Figure 1, regions 2, 3, 7, 8, and 9 are bounded by 3 arcs, hence $F_3 = 5$. While $F_2 = 1, F_4 = 1, F_5 = 1, F_6 = 1$. The map is regular, therefore

$$F_1 = 0 \tag{5}$$

The total number of regions is

$$F = \sum_{n=2,3,4\ldots} F_n \tag{6}$$

From (4), (5), and (6), we find

$$6 \sum_{n=2,3,\ldots} F_n - \sum_n nF_n = \sum_{n=2,3,\ldots} (6-n)F_n > 0 \tag{7}$$

Then

$$4F_2 + 3F_3 + 2F_4 + F_5 - F_7 - \ldots > 0 \tag{8}$$

Hence some $n < 6$ must exist for (8) to hold. Thus, any regular map on a surface of the Euler characteristic > 0 must have at least one region bounded by less than six arcs.

• PROBLEM 9-73

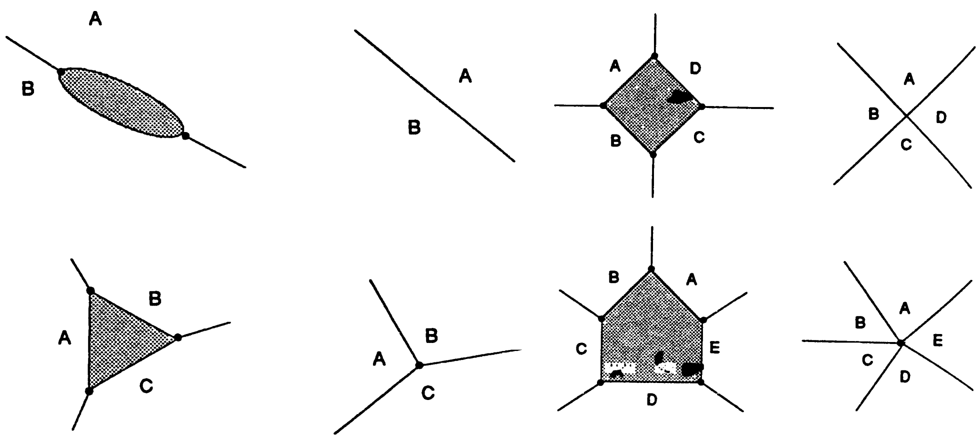
Using the results of Problem 9-72, prove the six color theorem.

SIX COLOR THEOREM

Any regular map on a surface of the Euler characteristic > 0 requires six colors at the most, if no neighboring regions are to be colored the same.

SOLUTION:

Obviously the theorem is true for $F \leq 6$. We shall prove that, if the theorem holds for F', then it holds for $F' + 1$. We proved in Problem 9-72 that at least one region exists which is bounded by less than six arcs, that is by two, three, four, or five arcs. Suppose this region shrinks to a point. The four possibilities are illustrated below.



For all cases with a contracted map, which contains F' regions the theorem holds by assumption. That is each of the contracted maps requires six colors. From the drawing, we see that when the region is restored there is a color available for it without the total of six colors being exceeded. Hence, if the theorem is true for F, then it is true for $F + 1$. Since it is true for $F \leq 6$, the proof is completed.

● PROBLEM 9–74

Any regular map on a surface of the Euler characteristic χ can be colored by at most β colors, where

$$\beta F > 6(F - \chi) \tag{1}$$

Show that at least one region must have a boundary consisting of less than β arcs.

SOLUTION:

For any regular map

$$V - E + F = \chi \tag{2}$$

$$2E \geq 3V$$

Thus

$$6(E - V) = 6(F - \chi) \geq 2E \tag{3}$$

Combining (1) and (3) we find

$$\beta F > 2E \tag{4}$$

Since the map is regular

$$\beta \sum_n F_n > \sum_n nF_n \tag{5}$$

$$\sum_n F_n(\beta - n) > 0\ ,\ n \geq 2 \tag{6}$$

From (6) we conclude that at least one region must have a boundary consisting of less than β arcs.

• PROBLEM 9–75

Determine the smallest integer β_0 satisfying

$$F\beta > 6(F - \chi) \tag{1}$$

when $F > \beta$. Consider the following values of χ : 0, 1, 2, –1, –2, ..., –10.

SOLUTION:

For $\chi = 0$, we have

$$\beta > 6. \tag{2}$$

The smallest integer β_0 satisfying (2) is $\beta_0 = 7$.

Let us write (1) in the form

$$\beta > 6(1 - \tfrac{\chi}{F}) \tag{3}$$

Then for $\chi = 1$, we obtain $\beta_0 = 6$. Similarly for $\chi = 2$, we find

$$\beta > 6 - \frac{12}{F} \tag{4}$$

The smallest integer β_0 satisfying (4) is $\beta_0 = 6$.

Note that the same values of β_0 were obtained from the six-color theorem.

We find:

$$\text{for } \chi = -1,\ \ \beta > 6 + {}^6/_F,\ \beta_0 = 7 \tag{5}$$

$$\text{for } \chi = -2,\ \ \beta > 6 + {}^{12}/_F,\ \beta_0 = 8$$

$$\text{for } \chi = -3, \;\; \beta > 6 + {}^{18}/_{F}, \beta_0 = 9$$

$$\text{for } \chi = -4, \;\; \beta > 6 + {}^{24}/_{F}, \beta_0 = 9$$

$$\text{for } \chi = -5 \;\; \beta > 6 + {}^{30}/_{F}, \beta_0 = 10$$

$$\text{for } \chi = -6 \;\; \beta > 6 + {}^{36}/_{F}, \beta_0 = 10$$

$$\text{for } \chi = -7 \;\; \beta > 6 + {}^{42}/_{F}, \beta_0 = 10$$

$$\text{for } \chi = -8 \;\; \beta > 6 + {}^{48}/_{F}, \beta_0 = 11$$

$$\text{for } \chi = -9 \;\; \text{we get } \;\beta_0 = 11$$

$$\text{for } \chi = -10 \;\; \text{we get } \;\beta_0 = 12 \tag{6}$$

We can summarize the results:

χ	β_0
2	6
1	6
0	7
−1	7
−2	8
−3	9
−4	9
−5	10
−6	10
−7	10
−8	11
−9	11
−10	12

● PROBLEM 9-76

From the inequality

$$\beta F > 6(F - \chi) \text{ for all } F > \beta \tag{1}$$

derive the formula for β_0 for negative values of χ (see Problem 9-75).

SOLUTION:

From (1) we have

$$\beta > 6(1 - \chi/_F) \tag{2}$$

Since $\chi < 0$, we can substitute for F the smallest admissible $\beta + 1$. Thus

$$\beta > 6(1 - \tfrac{x}{\beta+1}) \tag{3}$$

and

$$(\beta + 1)\,\beta > 6\beta + 6 - 6\chi \tag{4}$$

or

$$(\beta - \tfrac{5}{2})^2 > \tfrac{49}{4} - 6x \tag{5}$$

$$\beta > \sqrt{\tfrac{49}{4} - 6x} + \frac{5}{2} \tag{6}$$

The smallest integer β_0 satisfying (6) is

$$\beta_0 = \text{int}\left[\sqrt{\tfrac{49}{4} - 6x} + \frac{5}{2}\right] + 1 \tag{7}$$

where int $x = a$ means a is an integer and $x \geq a$.

For negative values of χ, we obtain:

χ	β_0
−1	7
−2	8
−3	9
−4	9
−5	10
−6	10
−7	10
−8	11
−9	11
−10	12
−11	12
−12	12
−13	13
−14	13
−15	13

● PROBLEM 9-77

Show that seven colors are necessary to paint a torus.

SOLUTION:

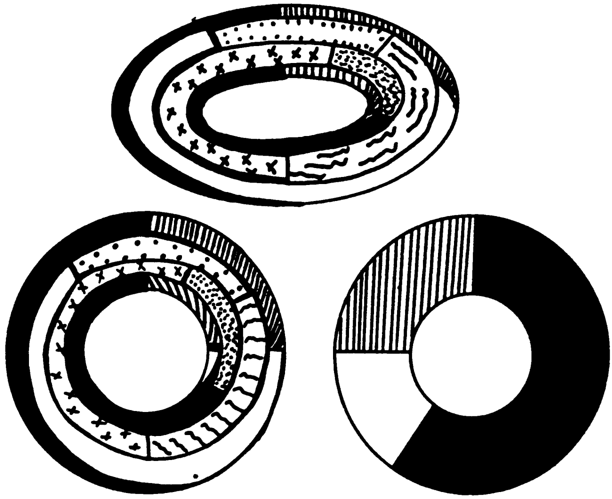


FIGURE 1

Figure 1 shows a torus divided into seven regions. Each country is a neighbor to the remaining six countries. We see that to paint a map on the torus, one needs at least seven colors. Hence the chromatic number β of the torus is

$$\beta \geq 7. \tag{1}$$

The Euler characteristic χ of the torus is $\chi = 0$. From Problem 9-75, we obtain

$$\beta \leq 7. \tag{2}$$

Hence, the chromatic number of the torus is 7.

● PROBLEM 9-78

Discuss the results of Problems 9-69, 9-71, 9-73, 9-75, 9-77.

SOLUTION:

A chromatic number, defined as the least number of colors required to color a map on any given surface, is a topological property of the surface. We tried to determine the chromatic number as a function of the Euler characteristic of the surface. Here are the results:

from Problem 9-69	for $\chi = 2$	$\beta \geq 4$
from Problem 9-73	for $\chi > 0$	$\beta \leq 6$
from Problem 9-75	for $\chi = 2$ or 1	$\beta \leq 6$
	for $\chi = 0$	$\beta \leq 7$
	for $\chi = -1$	$\beta \leq 7$
	for $\chi = -2$	$\beta \leq 8$
	for $\chi = -3$	$\beta \leq 9$
	for $\chi = -4$	$\beta \leq 9$
	for $\chi = -5$	$\beta \leq 10$
	for $\chi = -6$	$\beta \leq 10$
	for $\chi = -7$	$\beta \leq 10$
	for $\chi = -8$	$\beta \leq 11$
from Problem 9-77	for $\chi = 0$	$\beta \geq 7$

Note that the table in Problem 9-75 gives only the sufficient numbers of colors. Another problem is to determine the necessary number of colors.

It has been proven that for the surfaces of the Euler characteristic $\chi = 1$ or $\chi = 0$ or $\chi =$ an even negative integer, the values obtained in Problem 9-75 are necessary as well as sufficient. For those surfaces, the calculated values of β are the chromatic numbers. Hence

χ	**Chromatic Number**
1	6
0	7
-2	8
-4	9
-6	10
-8	11
-10	12

● PROBLEM 9-79

Suppose the world has the shape of a two-fold torus. What is the least number of colors needed to depict on the model of the world political situation?

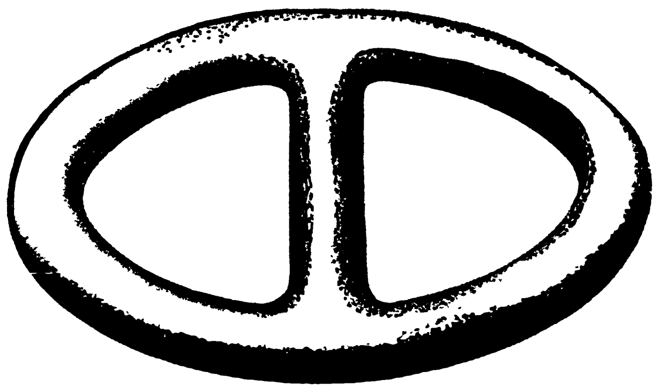


FIGURE 1

SOLUTION:

The Euler characteristic of a two-fold torus is $\chi = -2$. From the table (Problem 9-78), we find that chromatic number of a two-fold torus is $\beta = 8$. Hence, eight colors are needed to paint any map on the surface of a two-fold torus.

● PROBLEM 9-80

It has been proven that the maximum number of colors required for regular maps on a sphere or on a plane is six. Using Heawood's argument, show that this number can be reduced to five.

SOLUTION:

Any regular map on a sphere contains a region bounded by less than six arcs.

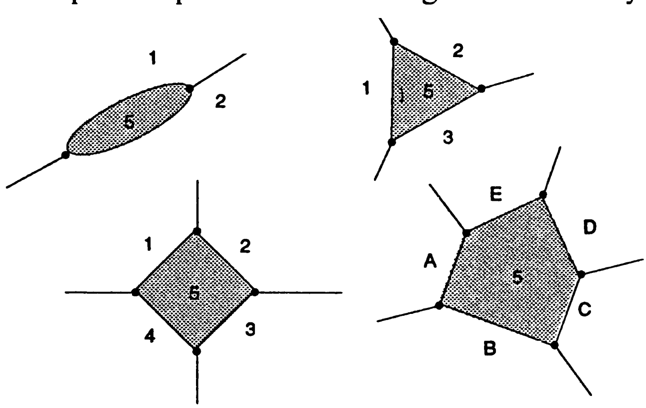


FIGURE 1

This region can always be painted with the color number five. It's obvious for the regions bounded by two, three, or four arcs.

For the region having five arcs as a boundary, some pair of the regions *A, B, C, D, E* must have no common boundary. Indeed, suppose *A* and *C* have a common boundary, then *B* has no common boundary with *D* (or *E*). Thus we need four colors to paint *A, B, C, D*, and *E* and the fifth one is available.

Suppose one arc is removed from the region bounded by two, three, or four arcs.

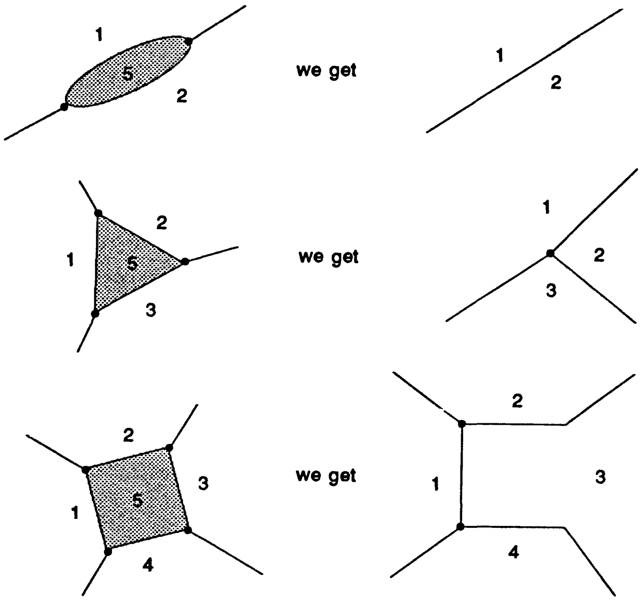


FIGURE 2

The number of regions decreases by one. If the maps in Figure 2 to the right can be colored with five colors, so can the original maps to the left.

If, for instance, *A* and *C* are of the same color, then by removing two arcs (Figure 3) we decrease the number of regions by two.

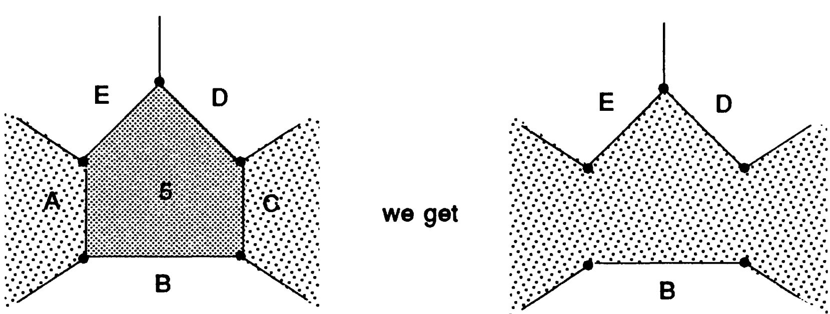


FIGURE 3

Again, if the map to the right (Figure 3) can be colored with five colors, then so can the original map. Gradually we reduce the number of regions until it is less than six. Since that can obviously be colored with less than six colors, the original map requires five colors or less. This result is known as Heawood's five color theorem.

● PROBLEM 9-81

How many colors are required to color every map on the plane surface?

SOLUTION:

Four! In 1976 the four-color problem was solved. For over a hundred years many mathematicians have tried in vain to prove (or disprove) this simple statement:

Four colors are required to color any map on the plane surface.

Attempting to find the solution, mathematicians like Heawood, Kempe and Birkhoff developed the new branch-graph theory, now used in arranging the airline routes and wiring diagrams.

In 1975 a group of scientists at the University of Illinois, using an IBM 360 computer, proved the four-color conjecture. The method used involved the reducibility of over 1,500 configurations.

It is amazing that the complete proof of this simple statement consists of hundreds of pages, miles of calculations and is impossible to carry out without the use of the computer.

Chances are that one day some bright teenager will find a short and elegant proof. It is also conceivable that no such proof is possible.

● PROBLEM 9-82

1. Which of the points *A, B, C,* and *D* lie inside the curve and which lie outside the curve?

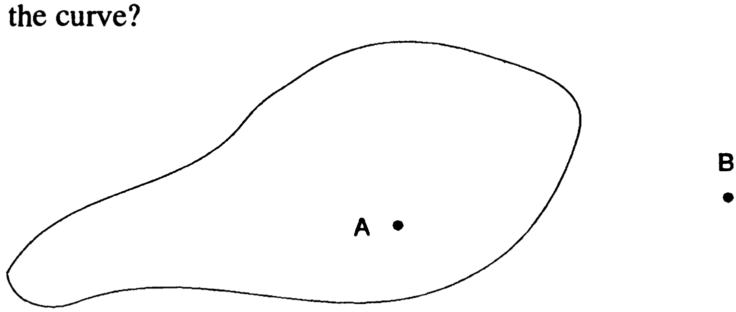


FIGURE 1

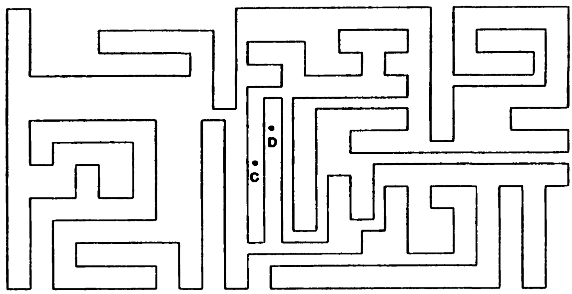


FIGURE 2

2. Why are both curves Jordan curves?

SOLUTION:

1. It is obvious that a triangle (or circle) separates a plane surface into an area inside and outside its perimeter. The same property is assumed for any continuous non-self-intersecting closed curve. For the curve of Figure 1, it is easy to see that A lies inside its perimeter and B outside its perimeter.

For more complicated curves, such as in Figure 2, it takes a while to determine what is inside and what is outside. In this case, C is outside and D is inside.

Even though the terms "inside" and "outside" are intuitively simple, they require a rigorous mathematical treatment.

2. By definition, any curve homeomorphic to a circle is called a Jordan curve.

• PROBLEM 9-83

Define and give an example of a polygonal path.

SOLUTION:

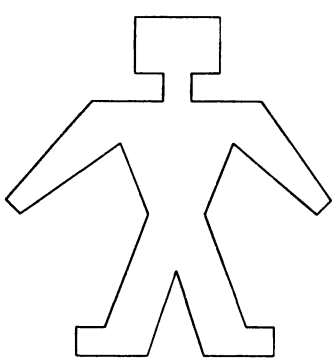


FIGURE 1

A polygonal path consists of n straight line segments joining n distinct points in a plane. Line segments can intersect only at their end points. Each line segment joins two points uniquely. The straight line segments are called the sides of the polygon. Polygonal paths determine the perimeter of the polygon. The figure depicts a polygonal path.

● PROBLEM 9-84

JORDAN CURVE THEOREM

On a plane or on the surface of a sphere, a Jordan curve separates the surface into two disjoint regions having the curve as a common boundary.

■

It is surprising that this theorem is obvious to almost everybody but it requires a fairly complex proof.

Prove the Jordan curve theorem for a polygonal path.

SOLUTION:

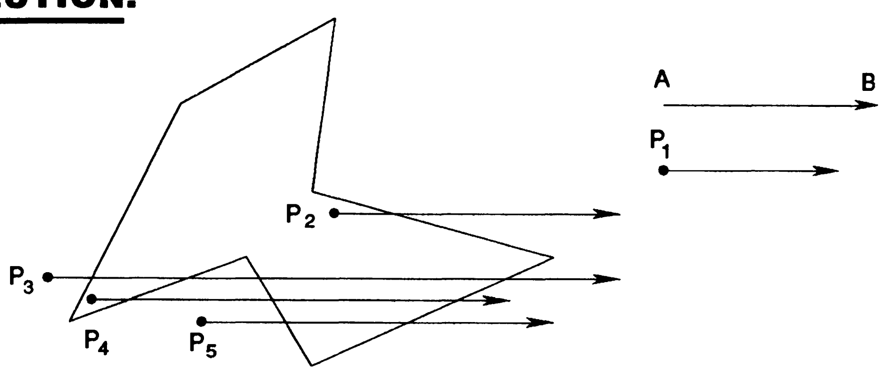


FIGURE 1

For a given polygonal path, it is always possible to find a line segment AB which is not parallel to any of the sides. This separates the plane (with the exception of the polygonal path) into two disjoint sets:

α - the set of points such that a ray from the point in the direction parallel to AB intersects the polygonal path an even number of times;

β - an odd number of times.

$$P_1 \in \alpha, \quad P_3 \in \alpha, \quad P_5 \in \alpha,$$

$$P_2 \in \beta, \quad P_2 \in \beta$$

from Figure 1.

Suppose a point moves along a line segment not intersecting a polygonal path and not parallel to AB, (Figure 2).

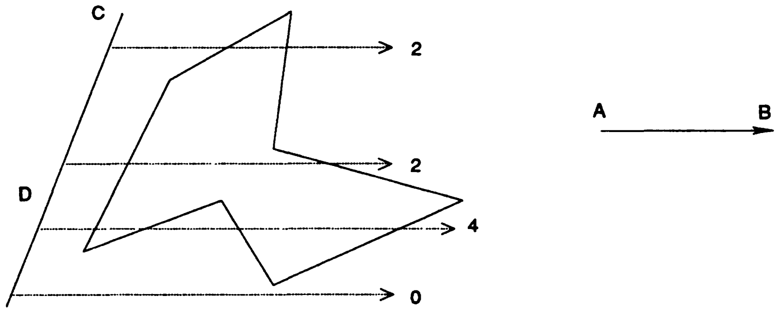


FIGURE 2

The number of intersections of the ray with the polygonal path always changes by a multiple of two. Thus, if any point belonging to α is joined to any point of β by a polygonal path, this new path must cross the old one an odd number of times. See Figure 3.

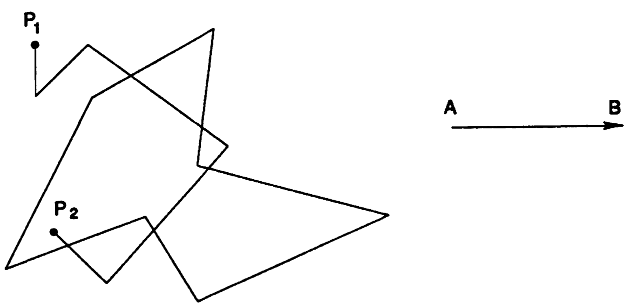


FIGURE 3

If both points P_1 and P_2 belong to the same subset (α or β), they can be joined by a polygonal path not intersecting the original polygonal path.

Thus it is possible to identify α as the set of points outside the polygonal path and β as the set of points inside.

● PROBLEM 9-85

Look again at the proof of the Jordan curve theorem for a polygonal path (Problem 9-84). Why does the same method applied to any Jordan curve lead to difficulties?

SOLUTION:

Since a polygonal path contains the finite number of sides of the polygon, it is easy to find a direction not parallel to any side.

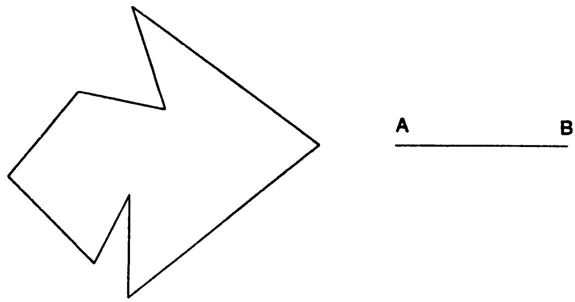


FIGURE 1

AB is not parallel to any side. For a closed curve it may no longer be possible to find a direction which is nowhere tangential to a curve. Obviously for a circle, no straight line in the plane exists, which is not parallel to some tangent to the circle. See Figure 2 where CD is parallel to AB.

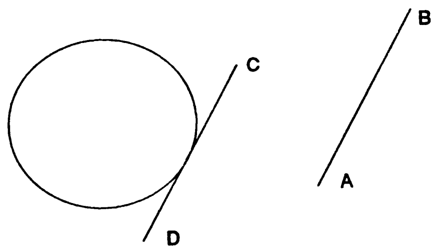


FIGURE 2

● PROBLEM 9-86

Another amazing statement in topology is that in a plane there are infinitely many simply connected bounded regions which all have the same boundary. Consider this puzzle for three regions:

FIGURE 1

There is an island. Within the first day you have to dig three canals:

from the sea, from the cold lake, and from the warm lake in such a way that the waters are separated and any point of the island is less than 1 mile from any water by the end of the day.

In the next half day the digging continues according to the same principles and each point has to be less than a $^1/_2$ mile from each kind of water. The work is carried out in intervals 1 day, $^1/_2$, $^1/_4$, $^1/_8$, $^1/_{16}$, ... of the day.

What happens at the end of the second day?

SOLUTION:

The island (or whatever is left of it) forms a closed set A nowhere dense. Each point of A is arbitrarily near to each kind of water.

Note that A is the common boundary of three regions: the sea, the warm lake, and the cold lake.

● PROBLEM 9–87

Rotate a disc and an annulus in its own plane about its own center. What point (or points) is always mapped into itself?

SOLUTION:

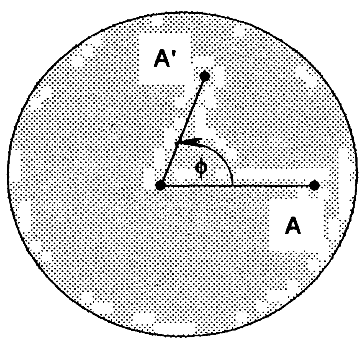


FIGURE 1

Let ϕ be the angle of rotation. The transformation is rigid, one-to-one and continuous. Each point A of the disc is mapped into A'. For ϕ not an integer multiple of 2π, there is one point only, which maps into itself, namely the center of the disc.

For the same rotation of an annulus there is no point which maps into itself.

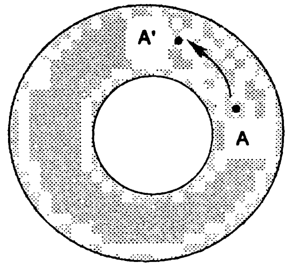


FIGURE 2

● PROBLEM 9-88

Explain the result of Problem 9-87 in terms of Brouwer's fixed point theorem.

SOLUTION:

BROUWER'S FIXED POINT THEOREM

For any continuous transformation of a disc to itself, there is at least one point which is mapped to itself.

■

Since rotation is a continuous transformation, at least one point of a disc is mapped into itself.

There is no continuous transformation f of a disc into itself

$$f: D \rightarrow D$$

such that, for every $x \in D$, $f(x) \neq X$.

Brouwer's fixed point theorem deals with the topological property, hence it holds for any region homeomorphic to a disc. It can be applied to various situations in life.

Consider for example a pool of oil spilled on the road.

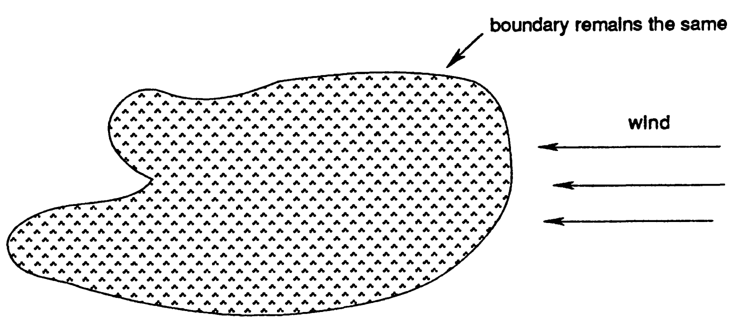
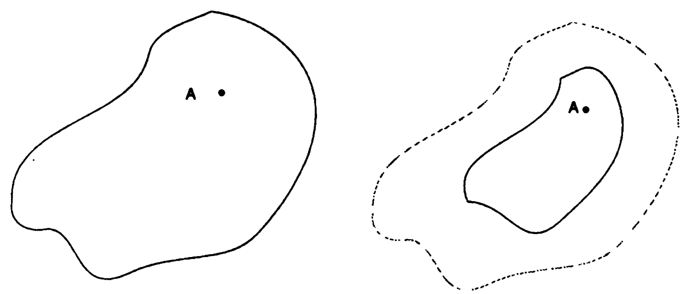


FIGURE 1

When the breeze blows, the surface of oil is moved. Assuming the boundary of oil remains the same and the oil is not "broken" in any way, there is at least one point where the oil remains in exactly the same place as it was before the wind began to blow.

● PROBLEM 9-89

A thin sheet of an elastic material is stretched as shown in Figure 1a.



(a) (b)

FIGURE 1

Show that when the material contracts so as to occupy only a part of its original area (Figure 1b), then there will always be a point A that occupies the same location before and after contraction.

SOLUTION:

Contraction is a continuous transformation, thus we can apply Brouwer's fixed point theorem.

We conclude that at least one point occupies the same place after contraction as before. Note that the same is true if we stretch out an elastic material.

● PROBLEM 9-90

Prove Problem 9-89 without direct reference to Brouwer's fixed point theorem.

SOLUTION:

Suppose the original region is *colored in the form of a chessboard* and

point P occupies P' after contraction. All squares can be grouped into three categories:

I. For every point belonging to the square, P' lies nearer the right side than this point.

II. For every point belonging to the square, P' lies nearer the left side than this point.

III. Squares that are neither I nor II.

It is easy to see that the squares of type I and II are never neighbors. Thus, square III goes from top to the bottom and a line can be drawn passing only through the squares of type III, which join the upper edge with the base.

Let us draw an arrow PP' at each point of this line. Since not all P' can lie higher than P not all arrows are directed upward. By the same token not all arrows are directed downward. Hence at least one point Q exists with a horizontal QQ'.

Since Q belongs to square III, there must be point R in this square with a vertical RR'.

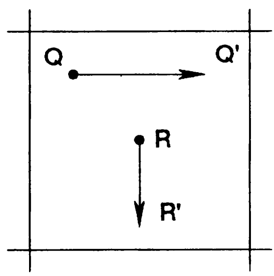


FIGURE 1

Since the square is small, such an abrupt change of direction is impossible (note that transformation is continuous) for a large PP'.

Hence, the arrow PP' for all points of the square must be small. When the number of squares increases, we reach the limit of point P_0, such that

$$P_0 P_0' = 0 \quad \text{that is,} \quad P_0 = P_0'$$

and P_0 is the fixed point of the transformation.

● PROBLEM 9-91

Here is a curious consequence of the Brouwer's fixed point theorem:

At any given time there are two points on the earth which are antipodes and which have the same temperature and the same air pressure.

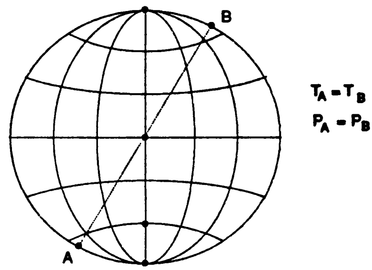


FIGURE 1

Explain why.

SOLUTION:

Suppose you have a sphere made of elastic material, for example, an inflatable ball. If the ball is folded and deflated so as to become flat, then there are two antipodes, A and B, which will lie one upon another.

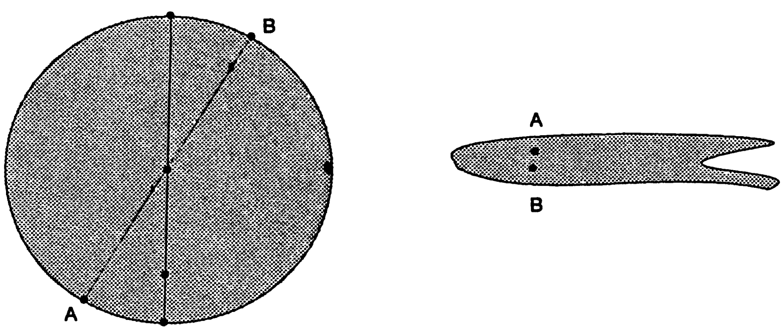


FIGURE 2

It can be proven by using Brouwer's fixed point theorem. Since both temperature and pressure are continuous, the conclusion follows.

● PROBLEM 9–92

Here is a fixed point theorem for one dimension:

If an interval is continuously transformed to itself, then there is at least one point of the interval which remains fixed.

■

One way to prove it is to divide the interval into small segments. List all complete segments for each division of the interval AB shown below.

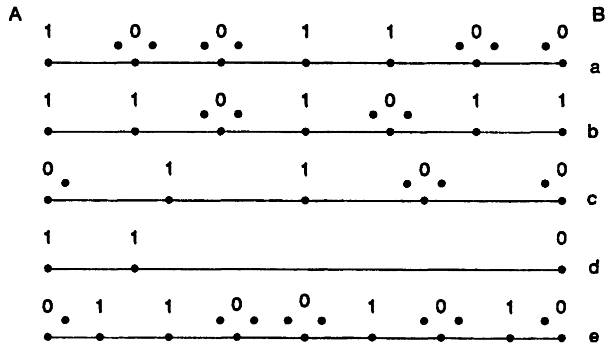


FIGURE 1

SOLUTION:

On the original segment AB we arbitrarily choose any number of points. All these points and the end-points A and B are labeled at random, 0 or 1.

On both sides of every 0, we put a dot. When a zero is at the end-point A or B, it gets one dot.

A small segment (i.e., one that does not include any subsegments) is called a complete segment when it has a 0 at one end and a 1 at the other.

For each division a to e, we list all small segments and circle the complete segments.

a. 10, 00, 01, 11, 10, 00

b. 11, 10, 01, 10, 01, 11

c. 01, 11, 10, 00

d. 11, 10

e. 01, 11, 10, 00, 01, 10, 01, 10

Each complete segment contains one dot, other segments contain no dots or two dots.

● PROBLEM 9–93

Show that if the end-points of the original line segment are labeled 0 and 1, then any division of this segment contains at least one complete segment.

SOLUTION:

Let n be the number of complete segments. Then the total number of dots is n plus some even positive integer

$$\# \text{ dots} = n + 2k.$$

On the other hand,

$$\# \text{ dots} = \# \text{ of 0's at the end-points} + 2 \cdot l$$

where l is the number of interval 0's.

We conclude that if the end-points of the original segment are labeled 0 and 1, then the number of complete segments must be odd. Thus, an original segment 01 (or 10) contains at least one complete segment.

● PROBLEM 9-94

By applying Problems 9-92 and 9-93, prove Brouwer's fixed point theorem for one dimension.

SOLUTION:

Suppose the original line segment, labeled 01, is continuously transformed into itself.

We label the points of the transformed segment 0, if their distance from the end-point 0 has not decreased and 1, if their distance from end-point 1 has not decreased.

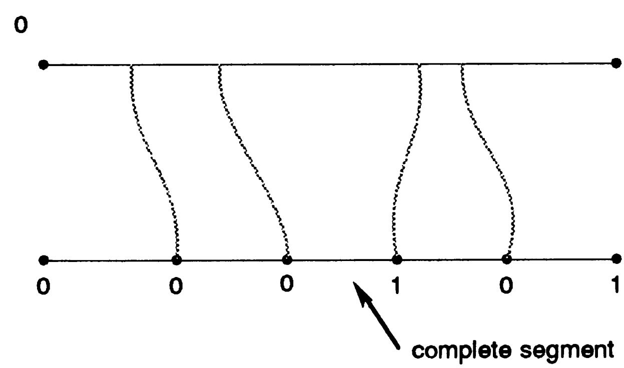


FIGURE 1

There must be at least one complete segment and since the labeling can go on infinitely, this segment can be made arbitrarily small.

At the limit, this segment tends to a single point, which is the fixed point we are seeking.

● **PROBLEM 9–95**

Prove Brouwer's fixed point theorem for two dimensions.

SOLUTION:

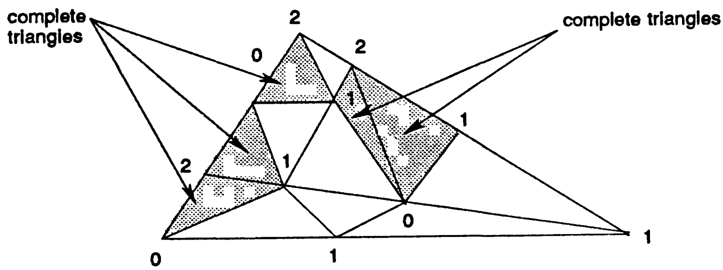


FIGURE 1

We shall use a method similar to the one-dimensional case. Consider a triangle arbitrarily subdivided into smaller triangles. The original triangle is labeled 012. The vertices lying on the side 01 are labeled 0 or 1 only, vertices on 02 are labeled 0 or 2 only and vertices on 12 are labeled 1 or 2 only.

A complete triangle is defined as a small triangle with vertices labeled 012.

This division of an original triangle contains at least one complete triangle. Next we continuously transform and divide the original triangle. Diminishing the complete triangle becomes the fixed point.

The above theorem is sometimes called Sperner's lemma.

In the same manner, we can prove Brouwer's fixed point theorem for any dimension.

● **PROBLEM 9–96**

Make a necessary cut of the cylinder (Figure 1) and draw the corresponding plane diagram.

SOLUTION:

Curves x and y are the boundaries of the cylinder. By cutting the cylinder along the line AB and then opening it, we obtain a rectangle. To ensure that the rectangle can be folded back, so as to give the cylinder we described its vertices; and use arrows to indicate that there is no twisting.

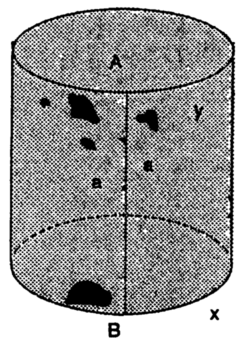


FIGURE 1

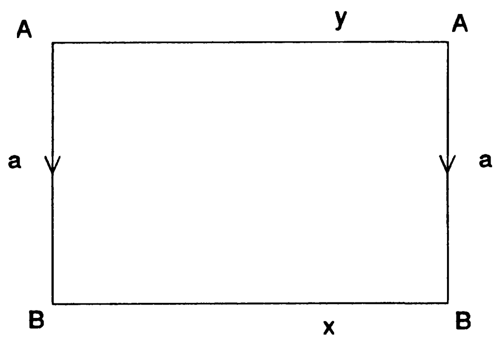


FIGURE 2

The diagram obtained as such is called a plane diagram.

● PROBLEM 9–97

For the torus depicted in Figure 1, make the needed cuts and sketch its corresponding plane diagram.

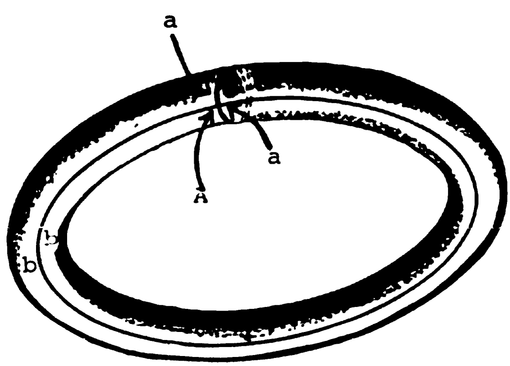


FIGURE 1

SOLUTION:

Two cuts, *a, b* are shown in Figure 1. The first cut, *a*, enables us to straighten the torus so as to obtain a cylinder.

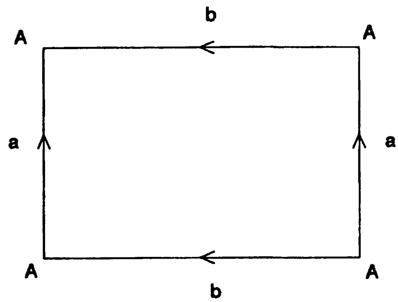


FIGURE 2

Here again arrows indicate that there is no twisting. Also note that all four vertices are labeled with the same letter.

● PROBLEM 9-98

Make the necessary cuts and draw a plane diagram (rectangle) of a sphere.

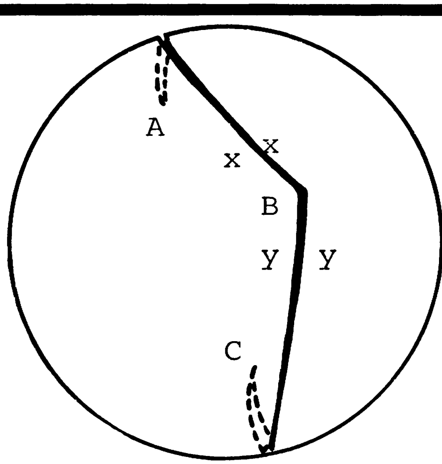


FIGURE 1

SOLUTION:

To obtain a plane diagram, we have to cut the sphere twice as shown in Figure 1. To obtain a rectangle, we have to deform the cuts. The plane diagram of the sphere is shown in Figure 2.

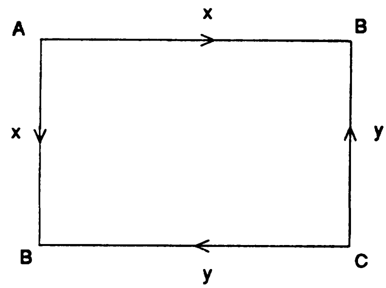


FIGURE 2

● PROBLEM 9-99

Sketch the plane diagrams representing a Möbius band and a Klein bottle.

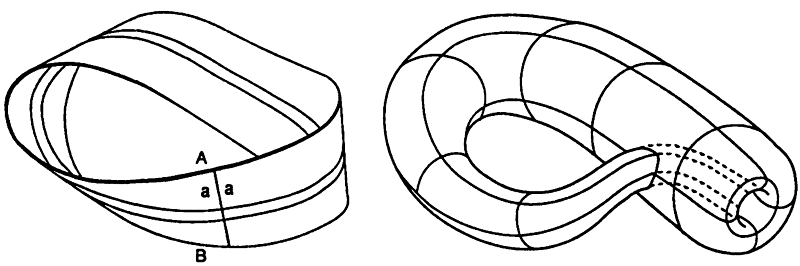


FIGURE 1

SOLUTION:

We cut a Möbius band along the line AB and then twist it (Figure 2). Note that we used two different letters, x and y, since the edges are not joined together.

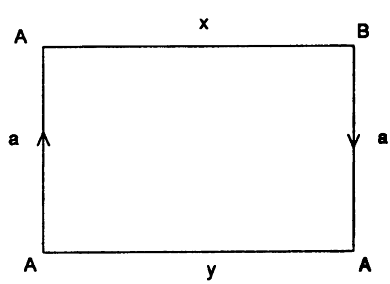


FIGURE 2

Note that from the plane diagram of a Klein bottle it is impossible to obtain a Klein bottle in three-dimensional space.

We have to make two cuts in order to obtain a plane diagram of a Klein bottle (Figure 3).

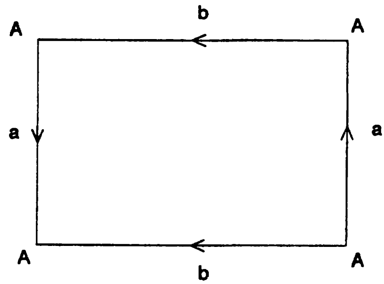


FIGURE 3

● PROBLEM 9–100

Draw the plane diagram of the real projective plane. Compare the plane diagrams of a Möbius band, Klein bottle and the real projective plane.

SOLUTION:

Compare these three diagrams:

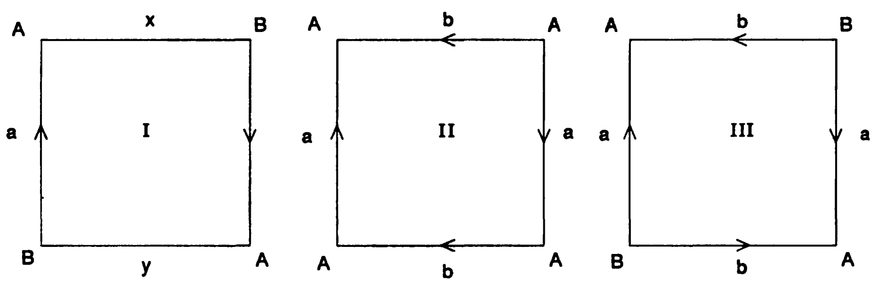


FIGURE 1

I is the plane diagram of a Möbius band. By adding arrows to avoid twisting, we obtain a diagram of a Klein bottle. If the opposite sides of the rectangle are to be joined together, then we should have to place the arrows in opposite directions as in diagram III. III is the plane diagram of the real projective plane, where each pair of opposite sides is joined in the opposite direction (with a twist).

The real projective plane can be represented as follows:

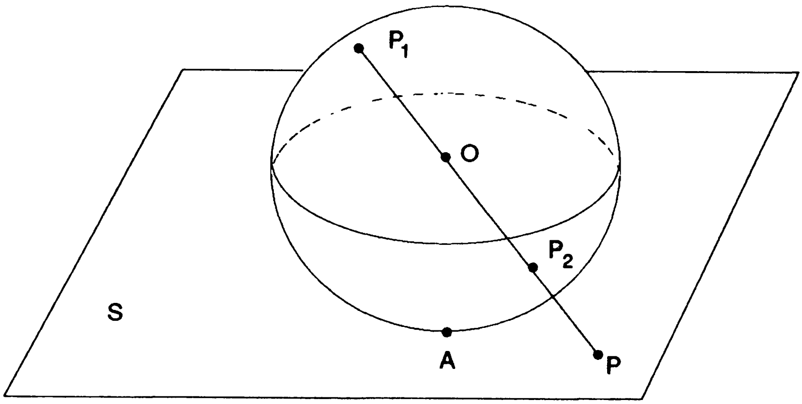


FIGURE 2

There is a sphere and a plane, which is tangential to the sphere at point A. Points P_1 and P_2 on the sphere are mapped to the single point P on the plane. Every great circle of the sphere is mapped to a line, with the exception of the great circle parallel to S. This circle is mapped into a line at infinity. The Euclidean plane, with the line of infinity added, is called the real projective plane.

● PROBLEM 9-101

From the plane diagrams, obtain the Euler characteristic χ of

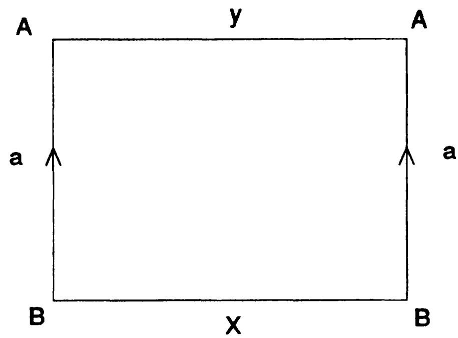
1. a cylinder
2. a one-fold torus
3. a sphere
4. a Möbius band
5. a Klein bottle
6. a real projective plane.

SOLUTION:

We shall apply the expression

$$\chi = V - E + F. \tag{1}$$

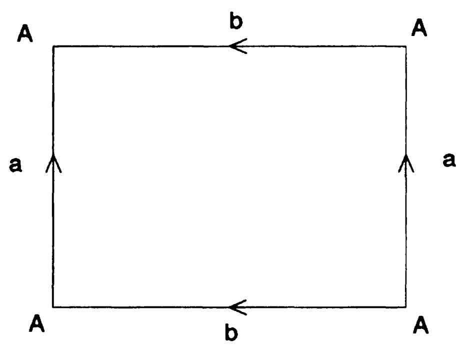
For a cylinder



there are two distinct vertices, $V = 2$, three distinct sides, a, x, y, $E = 3$, and one face $F = 1$. Thus

$$\chi = 2 - 3 + 1 = 0.$$

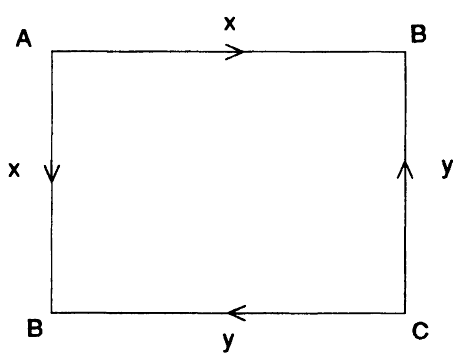
For a one-fold torus



$V = 1$, $E = 2$, $F = 1$, hence

$$\chi = 1 - 2 + 1 = 0.$$

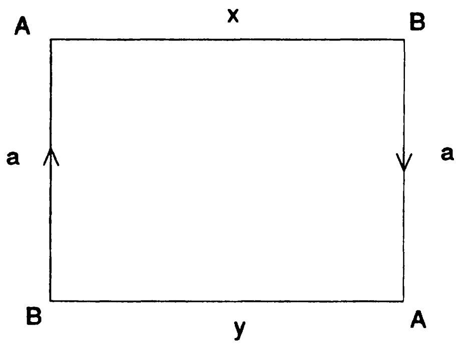
For a sphere



$V = 3$, $E = 2$, $F = 1$, hence

$$\chi = 3 - 2 + 1 = 2.$$

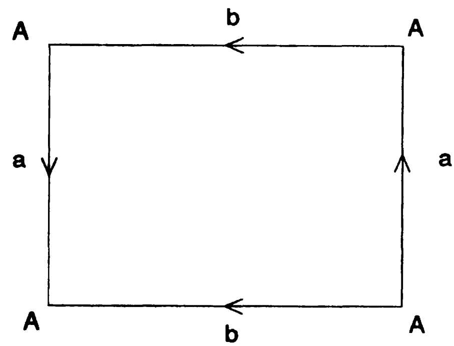
For a Möbius band



$V = 2, E = 3, F = 1$, hence

$$\chi = 2 - 3 + 1 = 0.$$

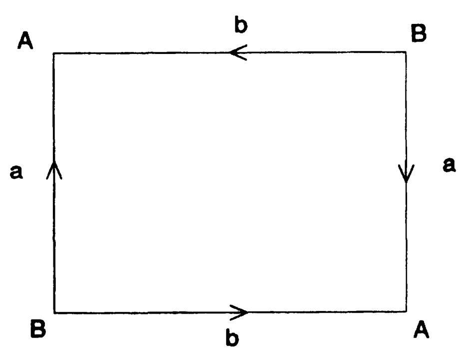
For a Klein bottle



$V = 1, E = 2, F = 1$, hence

$$\chi = 1 - 2 + 1 = 0.$$

For a real projective plane



$V = 2, E = 2, F = 1$, hence

$$\chi = 1.$$

● PROBLEM 9–102

We already proved that the maximum number of colors required for any map on the surface of a torus is seven.

By using a plane diagram of a torus, show that seven is a necessary number. Thus, prove that the chromatic number of a torus is seven.

SOLUTION:

By remembering that a plane diagram will be folded into a torus, we can color the rectangle as follows:

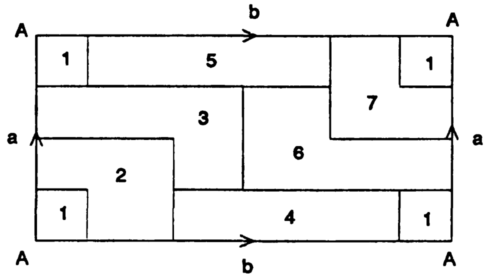


FIGURE 1

Note that when folded, each color borders with the remaining 6. This result is known as the seven color theorem for a one-folded torus.

● PROBLEM 9–103

Find the symbolic representation of a torus and a sphere.

SOLUTION:

The plane diagrams are

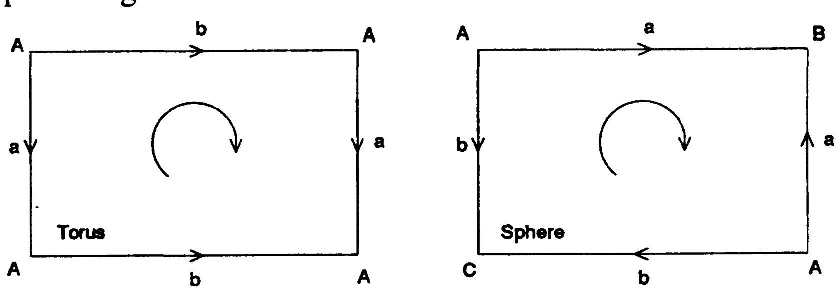


FIGURE 1

Let us establish the reference orientation as clockwise (we can choose counter-clockwise as well).

Starting from, for instance, the left upper corner and assigning + to each edge when the arrow moves clockwise or – when counter-clockwise we obtain

$$+b+a-b-a$$

for a torus. And

$$+a-a+b-b$$

for a sphere. Note that we can start from any vertex, hence, for a sphere equivalent representations are

$$-a+b-b+a$$

$$+b-b+a-a$$

$$-b+a-a+b$$

● PROBLEM 9-104

What is the Euler characteristic of a sphere with s holes?

SOLUTION:

The Euler characteristic of a sphere is $\chi = 2$.

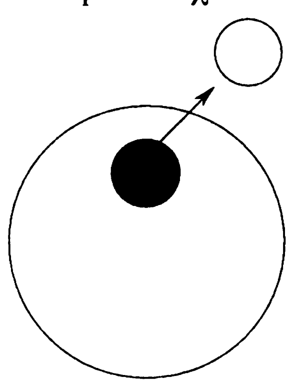


FIGURE 1

Suppose a hole is removed from a sphere (Figure 1). Then the number of the arc increases by one. The Euler characteristic

$$2 = \chi = V - E + F \tag{1}$$

decreases by one. Hence for a sphere with one disc removed

$$\chi = 2 - 1 = 1. \tag{2}$$

When two discs are removed

$$\chi = 2 - 2 = 0. \tag{3}$$

The sphere with two holes is homeomorphic to an open cylinder.

In general, a sphere with s holes has the Euler characteristic

$$\chi = 2 - s. \tag{4}$$

● PROBLEM 9-105

What is the Euler characteristic of a sphere with n handles and no holes?

SOLUTION:

A sphere with two handles and no holes (see Figure 1) is homeomorphic to a two-fold torus.

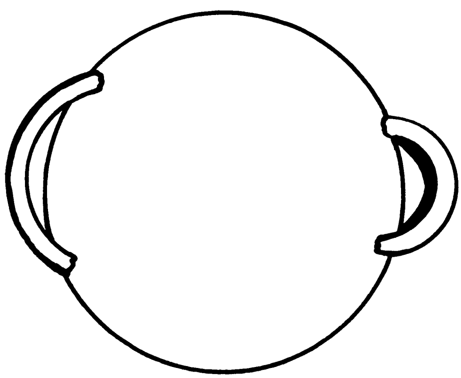


FIGURE 1

The Euler characteristic of a two-fold torus is -2. Hence, the Euler characteristic of a sphere with two handles is -2.

In general, a sphere with n handles and no holes left open is homeomorphic to an n-fold torus. The genus of the surface of an n-fold torus is n. But

$$\chi = 2 - 2g$$

Then

$$\chi = 2 - 2n$$

is the Euler characteristic of a sphere with n handles.

The Euler characteristic of a sphere with no handles is

$$\chi = 2 - 2 \cdot 0 = 2$$

with one handle

$$\chi = 2 - 2 = 0$$

with two handles

$$\chi = 2 - 4 = -2$$

etc.

● PROBLEM 9–106

Suppose a sphere with no holes and n handles is given with a map drawn on it. Since every region of the map must be simply connected, at least one arc must be drawn along every handle (Figure 1).

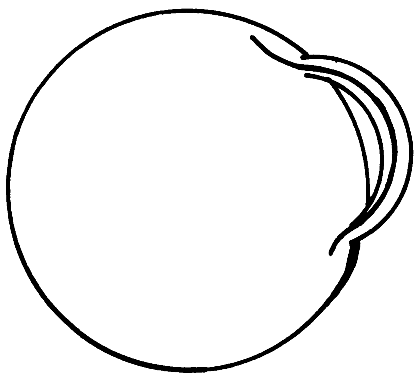


FIGURE 1

By using the procedure described below, determine the Euler characteristic of a sphere with n handles from

$$\chi = V - E + F.$$

SOLUTION:

We shall detach one end of each handle in order to obtain a sphere with n holes and n pipes, called cuffs.

I. **First Stage**

We have a map on a sphere with n handles and no holes, such that $\chi = V - E + F$.

II. Additional vertices are added to the map, one at every intersection of an arc and a boundary where a handle is to be disengaged. Such a boundary is now treated as an arc. Hence

$$V \text{ becomes } V + k$$
$$E \text{ becomes } E + 2k$$
$$F \text{ becomes } F + k$$

where k is the number of additional vertices to be added.

III. **Detachment**

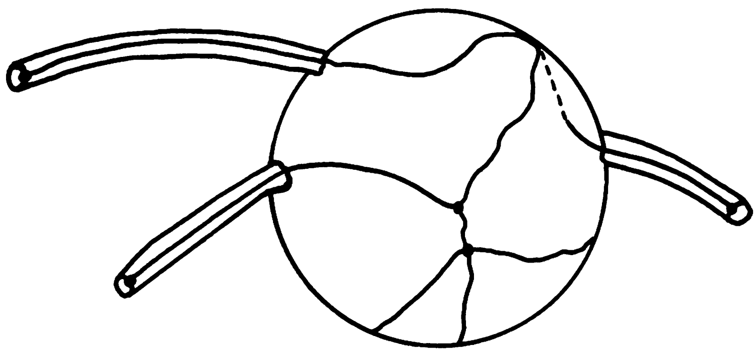


FIGURE 2

Figure 2 shows a sphere with 3 handles detached. In order to make a surface closed, we have to cover n holes and n open cuffs with $2n$ discs. When the handles are detached, k arcs and k vertices are added.

Now we have

$$V = 2k \text{ vertices, } E + 3k \text{ arcs, and } \mathrm{F} + K + 2n \text{ regions.}$$

The thereby obtained surface is homeomorphic to a sphere. Hence,

$$\chi = 2 = (V + 2k) - (E + 3k) + (F + k + 2n) \tag{2}$$

and

$$V - E + F = 2 - 2n \tag{3}$$

$$\chi = 2 - 2n. \tag{4}$$

Cf. Problem 9-106.

● PROBLEM 9–107

> The Euler characteristic of a cylinder is zero and of a disc is one. By attaching and removing various surfaces, find the Euler characteristic of a sphere with 7 handles and 4 holes.

SOLUTION:

It is easy to calculate the Euler characteristic from the formula

$$\chi = 2 - 2n - k \tag{1}$$

where n is the number of handles and k is the number of holes

$$\chi = 2 - 2 \cdot 7 - 4 = -16. \tag{2}$$

The other way is to start with a brand new sphere $\chi = 2$. Then make $2 \cdot 7 = 14$ holes for handles $\chi = 2 - 2 \cdot 7$, then 4 holes $\chi = 2 - 2 \cdot 7 - 4$, then put 7 cylinders to make handles, $\chi = 2 - 2 \cdot 7 - 4 + 7 \cdot 0$. Thus,

$$\chi = \chi\,[\text{sphere}] - \chi\,[\text{disc}] \cdot (2n + k) + \chi\,[\text{cylinder}] \cdot n =$$

$$= 2 - 14 - 4 = -16. \tag{3}$$

● PROBLEM 9-108

Generally, if n open surfaces $P_1, P_2, \ldots, P_n$ are joined together along the boundaries, then the Euler characteristic χ of the resulting surface is

$$\chi = \chi[P_1] + \chi[P_2] + \ldots + \chi[P_n]\,. \tag{1}$$

By applying (1), obtain the Euler characteristic of a sphere, putting together

1. n cylinders and two discs.

2. a sphere with k holds and k discs.

SOLUTION:

1. By putting n open cylinders, end to end, we obtain a single cylinder. By adding two discs, we obtain a surface homeomorphic to the surface of a sphere.

$$\chi\,[\text{sphere}] = \chi\,[\text{cylinder with two discs}] =$$

$$= n\chi\,[\text{cylinder}] + 2\chi\,[\text{disc}] = n \cdot 0 + 2 \cdot 1 = 2. \tag{2}$$

2. The Euler characteristic of a sphere with k holes is

$$\chi = 2 - k.$$

By covering k holes with k discs, we obtain a sphere. Hence

$$\chi\,[\text{sphere}] = \chi\,[\text{sphere with } k \text{ holes}] + \\ + k\chi[\text{disc}] = 2 - k + k = 2. \tag{3}$$

● PROBLEM 9–109

Determine the Euler characteristic of a cross-cap.

SOLUTION:

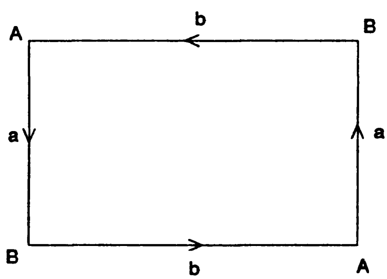


FIGURE 1

The plane diagram (Figure 1) of the real projective plane (see Problem 9-100) can be topologically deformed into a sphere with the appropriate hole shown in Figure 2.

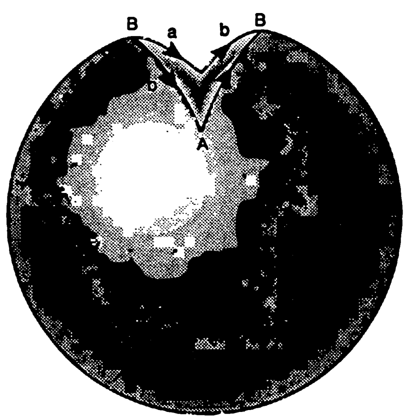


FIGURE 2

By closing the hole in such a way that the sides are joined according to the arrows, we obtain a closed surface intersecting itself (see Figure 3). The surface is one-sided. Points A and B are single; all other points of the segment AB are double points.

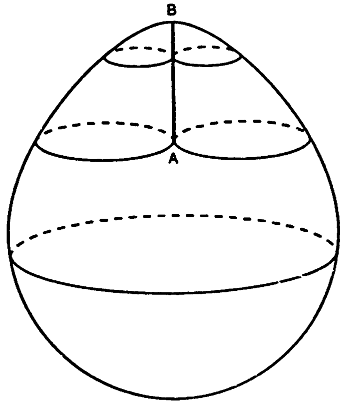


FIGURE 3

By removing the lower hemisphere which is homeomorphic, to a disc, we obtain a surface called a cross-cap (Figure 4).

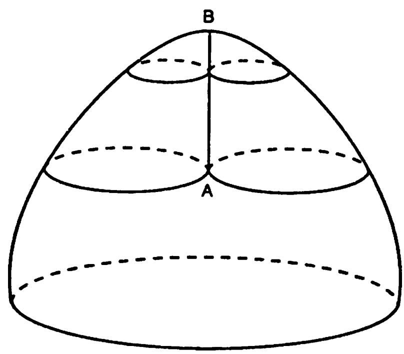


FIGURE 4

Hence,

$$\chi\,[\text{cross-cap}] = \chi\,[\text{real projective plane}] - \chi[\text{disc}] =$$

$$= 1 - 1 = 0.$$

The Euler characteristic of a cross-cap and a Möbius band are the same. Indeed, a Möbius band can be topologically deformed into a cross-cap.

● PROBLEM 9–110

Derive the formula for the Euler characteristic of a sphere with n handles, k holes and l cross-caps.

SOLUTION:

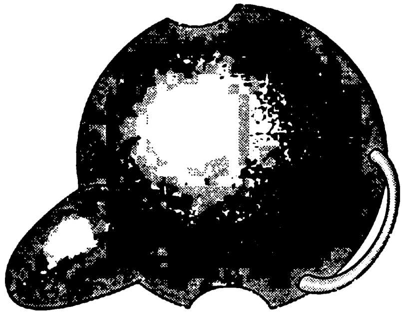


FIGURE 1

A sphere with one handle, one cross-cap and two holes is shown in the figure.

Note that each handle requires two holes and each cross-cap requires one hole. Thus, we have to make

$$2n + l + k$$

holes. The Euler characteristic of a cylinder is zero and of a cross-cap is also zero. Thus, the Euler characteristic of a sphere with n handles, l cross-caps and k holes is

$$\chi = 2 - (2n + l + k) = 2 - 2n - l - k.$$

● PROBLEM 9-111

Present in standard model form disc, sphere, cylinder, real projective plane, Möbius band, torus, two-fold torus and n-fold torus.

SOLUTION:

A surface described as a sphere with n handles, l cross-caps and k holes is said to be represented in standard model form.

	n-handles	**l cross-caps**	**k holes**	χ
sphere	0	0	0	2
disc	0	0	1	1
cylinder	0	0	2	0
real projective plane	0	1	0	1
Möbius band	0	1	1	0

torus	1	0	0	0
two-fold torus	2	0	0	-2
n-fold torus	n	0	0	$2-2n$

● PROBLEM 9-112

Determine the rank and Euler characteristic of a torus with a hole.

SOLUTION:

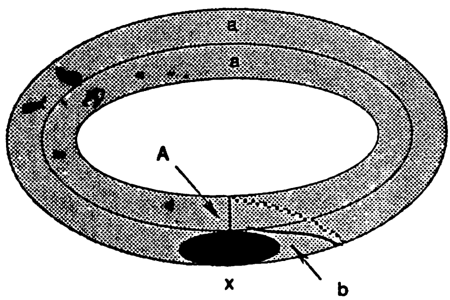


FIGURE 1

The rank of an open surface is defined as requiring the least number of cuts to make the surface homeomorphic to a disc.

From Figure 1, we see that two cuts are required to make a torus with a hole homeomorphic to a disc. The cut surface can be deformed to yield a plane diagram (Figure 2) with the symbolic notation

$$\pm x + a + b - a - b.$$

The torus with a hole has 1 handle, 1 hole and 0 cross-caps. Hence

$$\chi = 2-(2n+k+l) = 2-2-1-0 = -1.$$

The same value can be obtained from a plane diagram

$$\chi = V - E + F = 1 - 3 + 1 = -1.$$

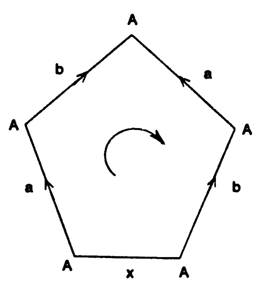


FIGURE 2

Find the rank and Euler characteristic of a sphere with one cross-cap and one hole.

SOLUTION:

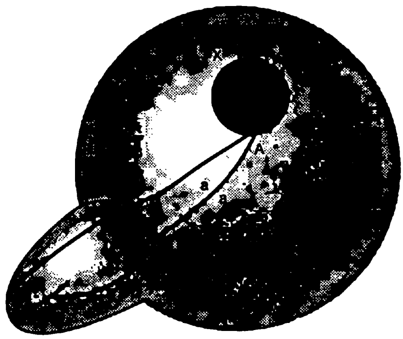


FIGURE 1

This is an open surface. It requires one cut (Figure 1) to become homeomorphic to a disc.

The rank of a sphere with one cross-cap and one hole is one. The surface obtained can be deformed to give a triangle.

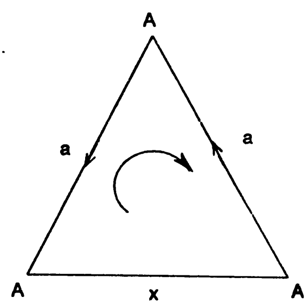


FIGURE 2

Its symbolic representation

$$\pm x - a - a$$

The Euler characteristic

$$\chi = 2 - (2n + k + l) = 0$$

or from the plane diagram

$$\chi = V - E + F = 1 - 2 + 1 = 0.$$

● PROBLEM 9-114

Why is predicting the physical and chemical properties of chemical substances before they are synthesized so important?

SOLUTION:

More than seven million different molecules have been synthesized. At least some of their properties are known with fairly good accuracy. In some sense it forms a basis for future research. The cost of obtaining new molecules can sometimes be enormous. It can happen that the final product turns out to be a disappointment. A new molecule is not exactly what we have been looking for.

Hence, it is important, not only from the scientific point of view, to be able to predict the properties of chemical substances before they are synthesized.

● PROBLEM 9-115

Describe briefly the topological method of making chemical predictions.

SOLUTION:

The heart of this method is the topology of individual molecules. We neglect the nature and lengths of the chemical bonds holding the atoms of a molecule together. We take into account the number of atoms in the molecule and how the atoms are interconnected within the molecule (single straight chain, chain with branches, several chains, a ring, etc.).

As a first step we take a small number of known molecules, for instance, $A_1, A_2, \ldots, A_k$. Then we establish a procedure which converts the topological structure of each molecule A into a single, characteristic number α called index.

$$A_1 \rightarrow \alpha_1$$

$$A_2 \rightarrow \alpha_2$$

$$\vdots$$

$$A_k \rightarrow \alpha_k$$

Suppose we are interested in a certain chemical property, say the boiling point. For each $A_1, A_2, \ldots, A_k$ we determine its boiling point $T_1, T_2, \ldots, T_k$. By using the Cartesian coordinates on one axis, we mark the value of the index

and on the other, the value of the chemical (physical) property.

Thus, we obtain a plot which can be used to predict the future.

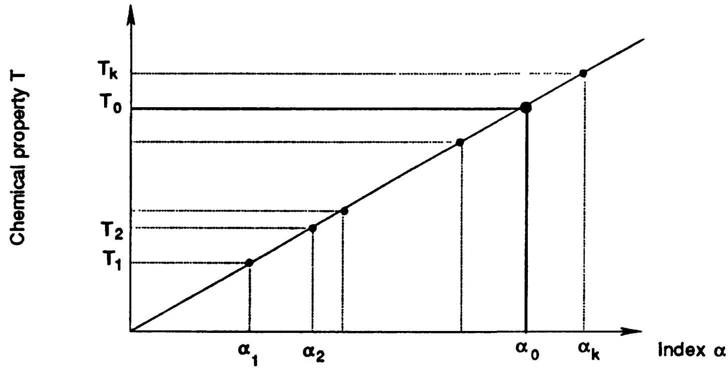


FIGURE 1

Suppose we want to know the boiling point of a molecule whose index is α_0. From the plot, we obtain T_0.

● PROBLEM 9-116

What is a chemical graph?

SOLUTION:

A chemical graph is a drawing in which the atoms of the molecule are depicted as points and the bonds linking the atoms are depicted as straight lines. The length of lines and the angles are immaterial.

Thus, graphs represent structures in an abstract manner. Graph theory, a mathematical discipline, studies graphs in detail.

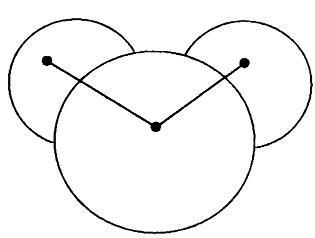


FIGURE 1–Graph of H_2O

Generally, in chemical graphs the hydrogen atoms are omitted.

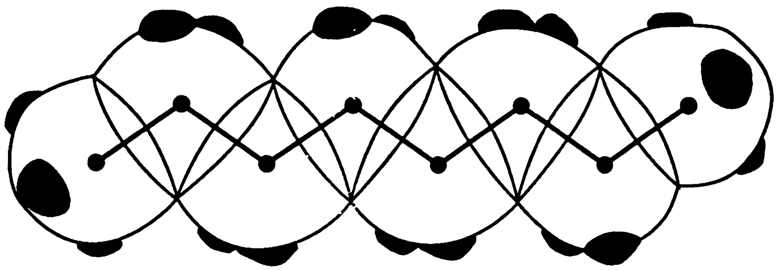


FIGURE 2-Graph of an n-octane.

Large balls depict carbon atoms and small balls depict hydrogen atoms.

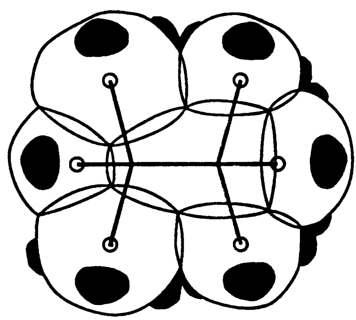


FIGURE 3–The structure of 2, 2, 3, 3,-tetramethylbutane.

The graph is shown inside with the hydrogen atoms omitted.

● PROBLEM 9–117

Describe how the carbon number is determined.

SOLUTION:

Once the chemical graph of a molecule has been drawn, it is fairly easy to compute its characteristic number - index. Such a number is known as a graph invariant. One of the most simple graph invariants is the vertex number, which chemists refer to as the carbon number. It is the number of vertexes in the graph. For hydrocarbon molecules (i.e., molecules consisting only of hydrogen and carbon atoms) it is the number of carbon atoms. It should be remembered that the carbon number has low discriminating power because many different molecules can have the same carbon number.

● PROBLEM 9-118

Describe the Wiener index and its applications.

SOLUTION:

The carbon number is a useful index for analyzing straight-chain molecules for branched molecules. The Wiener index is more appropriate. This index uses the notion of distance between two vertices, which is defined as follows: The distance between any two vertices is equal to the number of edges traversed while taking the shortest route within the graph between these vertices.

The Wiener index of a molecule is equal to the sum of the distances between all pairs of atoms in the molecule.

Consider for example isopentane.

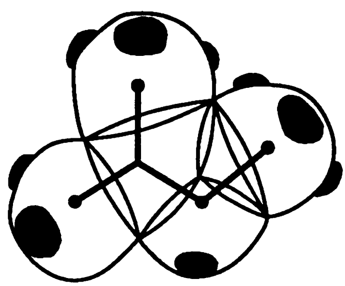
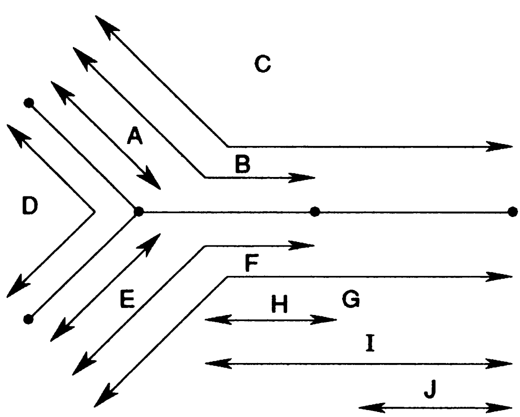


FIGURE 1

Its graph and distances are shown in Figure 2.



A	1
B	2
C	3
D	2
E	1
F	2
G	3
H	1
I	2
J	1
	18

WIENER INDEX

FIGURE 2

● PROBLEM 9–119

Show how the Wiener index can be applied to determine the chemical or physical property of a molecule that has not yet been synthesized.

SOLUTION:

The Wiener index as described in Problem 9-118 converts the topological structure of a molecule into a single number-index. It has been determined that the Wiener index correlates with many physical properties.

First, we establish the molecular structure of some existing molecules and draw their graphs.

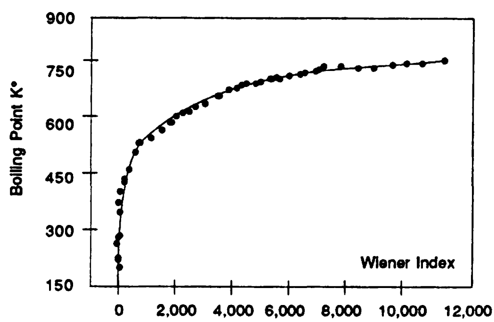


FIGURE 1

Suppose we are interested in the boiling point. For each molecule, we know its Wiener index and its boiling point. Thus, obtained points yield a fairly smooth curve (see Figure 1). The correlation is more evident when we use the logarithms of the Wiener index and boiling point (see Figure 2).

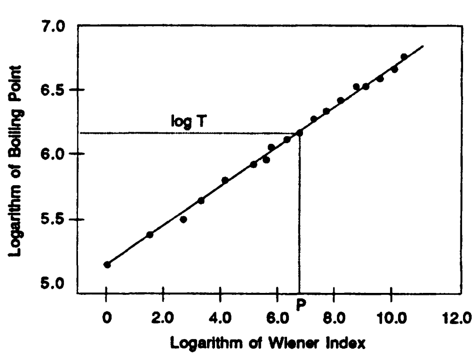
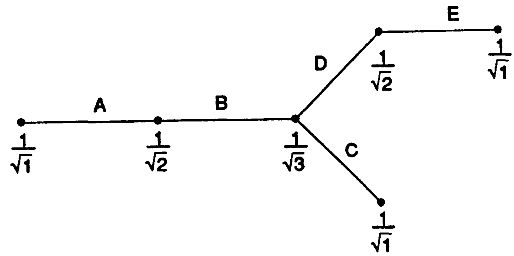


FIGURE 2

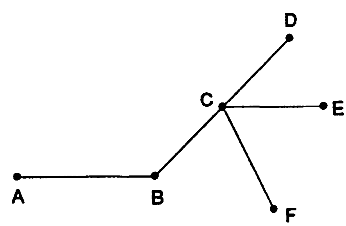
Suppose we want to synthesize a molecule whose logarithm of the Wiener index is P. From the plot we find the logarithm of its boiling point.

Compute the molecular-connectivity index of a molecule whose graph is shown below:



SOLUTION:

The molecular-connectivity index is the most universal index devised so far. It depends on the topological concept of degree. The degree of any vertex is equal to the number of other vertices it is attached to.



Hence the degree of A is 1, of B is 2, of C is 4, of D is 1, of E is 1 and of F is 1.

Each edge has a value: the product of the reciprocals of the square roots of the degrees of the vertices it joins.

$$\text{Molecular - connectivity Index} = \sum \frac{1}{\sqrt{N_i} \times \sqrt{N_j}}$$

where the sum is taken over all the edges.

Hence, the molecular-connectivity index for our molecule is

$$\left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{1}}\right) + \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{2}}\right) +$$

$$+ \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{1}}\right) = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} = 2.80806$$

● PROBLEM 9-121

In some cases, better correlations are obtained by considering only the fragments of larger molecules. Compute the molecular-connectivity index for the fragments of molecules indicated by dotted lines.

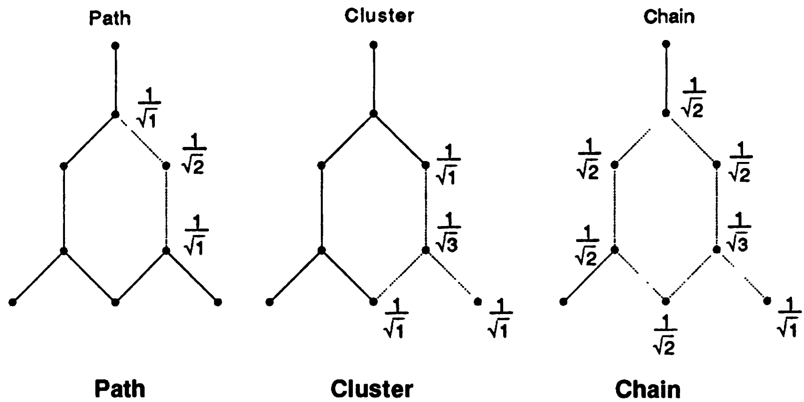


FIGURE 1

SOLUTION:

From the definition of the molecular-connectivity index, we obtain:

Path

$$\left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{1}}\right) = 1.4142$$

Cluster

$$\left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) = 1.73205$$

Chain

$$4 \times \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + 2 \times \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) = 3.39384$$

● PROBLEM 9-122

Compute the Balaban centric index for a molecule whose graph is shown below.

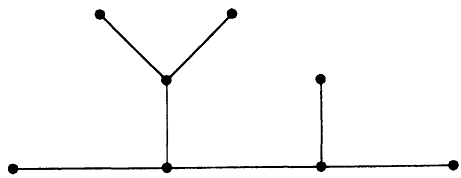


FIGURE 1

SOLUTION:

This index emphasizes the degree of branching in a molecule. All vertices that are connected to only one other vertex are counted and removed from the molecule's graph. The number of vertices removed at each step is squared and added to a running total. The procedure is repeated until all vertices are removed.

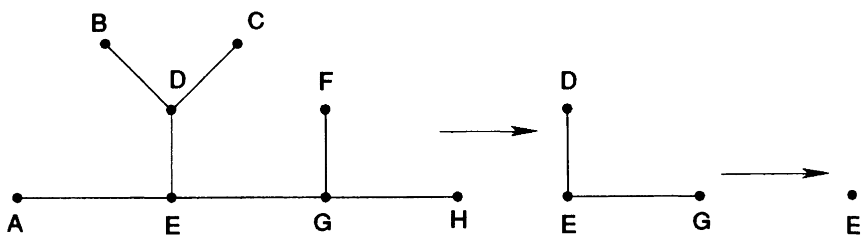


FIGURE 2

As a first step we remove vertices A, B, C, F, and H and get 5^2. Then we remove D and G and obtain 2^2. Finally, we remove E and obtain 1^2.

Thus, the index is $5^2 + 2^2 + 1^2$.

● PROBLEM 9–123

List a few applications of molecular-connectivity index.

SOLUTION:

The molecular-connectivity index has numerous and diversified applications. It is used in developing new drugs and in predicting the taste and smell of new substances. This index correlates well with such physical properties as the boiling point, density, heat of vaporization and solubility in water. It is known that many biological responses are launched when an appropriate stimulating molecule docks with a receptor on the surface of a cell. Very

often the specific shape of the molecule is less important than its volume and surface area. Molecular-connectivity indices correlate well with volume and surface area. Thus, the molecular-connectivity indices can predict the ability of molecules to act as anesthetics, hallucinogens or narcotics.

It is also possible to predict the smell of molecules as well as whether a molecule will taste bitter or sweet.

● PROBLEM 9-124

Describe the applications of molecular-connectivity indices in environmental studies.

SOLUTION:

It is a very costly and time-consuming process to determine how fast pollutants spread into the environment. Here, molecular-connectivity indices are very helpful.

The indices correlate well with the ability of many pollutants to spread within the air, water or soil. The relationship exists between the indices and the tendencies of substances to concentrate within living organisms. To test these properties without the indices is extremely difficult and expensive. That is why the U.S. Environmental Protection Agency applies the indices to predict the toxic potentiality of unknown or untested pollutants.

● PROBLEM 9-125

What does the octane number of a fuel describe? What methods are used to predict it?

SOLUTION:

In general, the octane number describes the efficiency with which a fuel burns, that is, its tendency not to "knock." The fuel is mixed with oxygen and then ignited by a spark.

A "knock" appears when during compression, oxygen atoms combine with fuel before it has been ignited. Usually the octane number is determined under standardized conditions in a test engine.

Straight-chain molecules tend to "knock" more than branched molecules.

Attempts to establish the correlation between the tendency of the fuel to "knock" and its Wiener index yielded only fair results.

An index introduced by A. Balaban, called the centric index, emphasized the branching of molecules. It correlated very well with the octane number of

hydrocarbon molecules.

We defined the centric index in Problem 9-123.

● PROBLEM 9-126

An index called hydrogen-deficiency index was devised to predict the amount of soot produced by burning hydrocarbon molecules. Define this index.

SOLUTION:

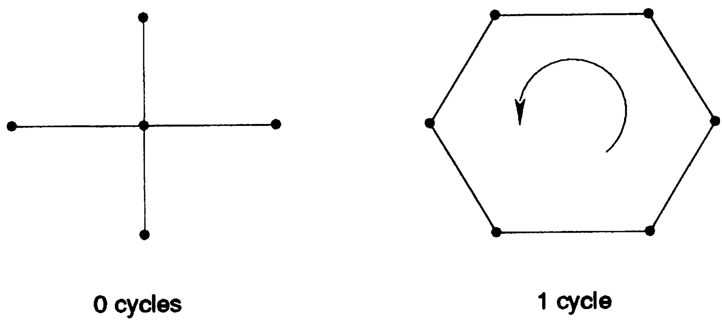


FIGURE 1

The hydrogen-deficiency number is equal to the number of independent cycles, or nips, and double bonds in a molecule.

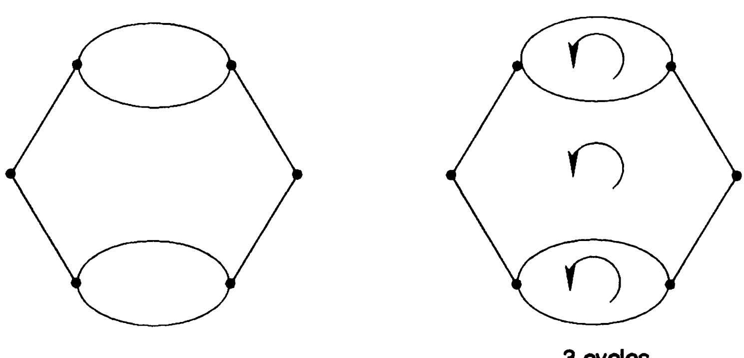


FIGURE 2

The hydrogen-deficiency number combined with the averaged-distance-sum connectivity index yields an index which provides a very good correlation with the soot production for almost 100 hydrocarbon molecules.

● PROBLEM 9-127

Describe how the cancer-causing tendency of a molecule (carcinogenicity) can be predicted by means of topological methods.

SOLUTION:

The task of finding a correlation between carcinogenic behavior of molecules and topological indices is an extremely difficult one.

First of all, the experimental measurements are not necessarily accurate or sufficient. Then the growth of cancer is a process consisting of many stages.

Thus, the index should take into account not only the original molecules but also the molecules produced during successive stages. So far the best results were achieved by William C. Herndon and László van Szentpály. They introduced the combination of simple indices to predict the carcinogenicity of polycyclic aromatic hydrocarbons. Since the molecules, below and above certain size limits, are not carcinogenic, the final index includes the carbon number and the square of the carbon number (which depends on the size of the molecules). Another aspect is that certain regions of hydrocarbons are more important (more carcinogenic) than other regions.

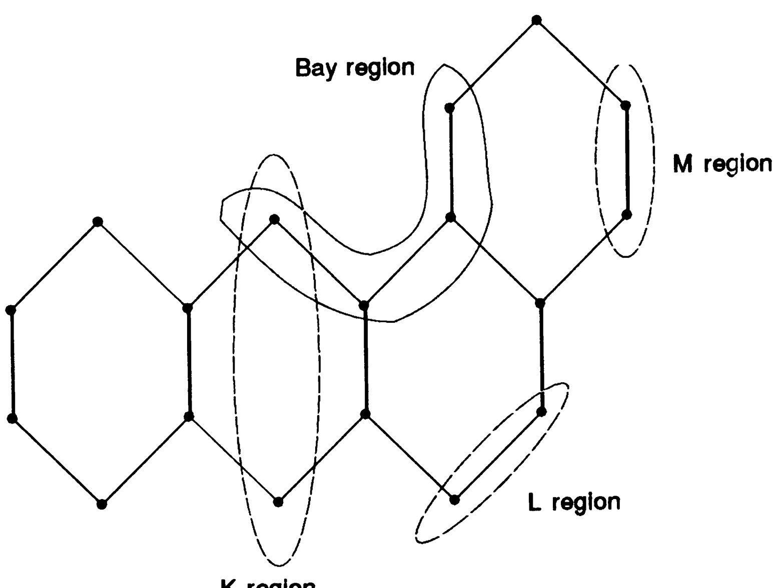


FIGURE 1

The Bay, *L* and *M* regions must be fairly active chemically, while the *K* region must remain fairly inactive.